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Elliptic curve 2y^2=x^3+x over field size 8^91+5

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Abstract

Multi-curve elliptic curve cryptography with curve 2y^2=x^3+x/GF(8^91+5) hedges a risk of new curve-specific attacks. This curve features: isomorphism to Miller's curve from 1985; low Kolmogorov complexity (little room for embedded weaknesses of Gordon, Young--Yung, or Teske); similarity to a Bitcoin curve; Montgomery form; complex multiplication by i (Gallant--Lambert--Vanstone); prime field; easy reduction, inversion, Legendre symbol, and square root; five 64-bit-word field arithmetic; string-as-point encoding; and 34-byte keys.

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1. Introduction

Elliptic curve cryptography (ECC) is now part of several IETF protocols.

Multi-curve ECC can mitigate the risk of new curve-specific attacks on ECC.

> Note: The risk of new curve-specific attacks is arguably smaller than the risk of quantum attacks breaking all current curves in ECC. The first risk can be addressed by combining ECC with post-quantum cryptography (PQC). Multi-curve ECC would then be added to an ECC+PQC combination. The extra cost of multi-curve over single-curve ECC would less noticeable in an ECC+PQC combination than it would be in an ECC-only implementation.

This document aims to contribute to multi-curve ECC by describing how to use the curve

> 2y^2=x^3+x / GF(8^91+5)

for elliptic curve Diffie--Hellman (ECDH).

Appendix A expands on why and when 2y^2=x^3+x/GF(8^91+5) is useful in multi-curve ECC.

2. Requirements Language (RFC 2119)

The key words "MUST", "MUST NOT", "REQUIRED", "SHALL", "SHALL NOT", "SHOULD", "SHOULD NOT", "RECOMMENDED", "MAY", and "OPTIONAL" in this document are to be interpreted as described in RFC 2119 [BCP14].

3. Use ONLY in multi-curve ECC

An implementation using curve 2y^2=x^3+x/GF(8^91+5) in elliptic curve cryptography SHOULD use it multi-curve ECC in a combination with more established, less special, curves, such as Curve25519 or NIST P-256, or even Brainpool curves.

The curve 2y^2=x^3+x/GF(8^91+5) belongs to a very special class. Miller in 1985 suggested to exercise prudence around similarly special curves in ECC. This document continues to heed Miller's advice, by considering 2y^2=x^3+x/GF(8^91+5) riskier than less special curves, and hence advising to use it only a second layer of defense in ECC.

4. Representations

Interoperable communication on most systems involves byte strings. Elliptic curve cryptography needs to use byte strings to work on these systems. Different curves use byte strings in different ways, due to differences in the curves such as the number of points on the curve.

This section specifies a default compressed representation of points of 2y^2=x^3+x/GF(8^91+5) as 34-byte strings. The compressed representation SHOULD be used for communication whenever byte string representations are needed. The compressed representation MUST be used when computing an ECDH shared secret.

An OPTIONAL uncompressed representation as 70-byte strings is also specified. The uncompressed representation MAY only be used for systems that optimize speed, and are willing to send extra an 35 bytes. The uncompressed representation MUST NOT be used to represent an ECDH shared secret.

An OPTIONAL representation of scalar multipliers, such as secret keys, as 34-byte strings is also provided. This representation is used for secrets, and is not for public communication. Different components of a single ECC system MAY use this representation to work together.

An OPTIONAL representation of 34-byte strings as points is also described. This MAY be used in niche applications, such as in deriving a point from the hash of a password, or in steganography, to hide the presence of ECC, by choosing special points such that represented in a way indistinguishable from uniformly random byte strings. This uses the Elligator i method, a variant of the Elligator 2 method.

4.1 Representation points in 34 bytes (compression)

Elliptic curve cryptography uses points on a curve for its public keys and for it raw shared secret Diffie--Hellman keys.

Abstractly, points are mathematical objects. For curve 2y^2=x^3+x, a point is either a pair (x,y), where x and y are elements of mathematical field, or a special point O (whose x and y coordinates may be deemed as infinity).

> Note: The special point O should never be used as a key, in practice. In theory, point O is needed for the points to form a mathematical group.

For curve 2y^2=x^3+x/GF(8^91+5), the coordinates x and y of the point (x,y) are integers modulo 8^91+5, which can be represented as integers in the interval [0,8^91+4].

> Note: An implementation will often internally represent the x-coordinate as a ratio [X:Z] of field elements. Each field element has multiple such representations. The ratio [x:1] can viewed as the normalized representation of x. (Infinity can be then represented by [1:0].)

This draft specifies an encoding of finite points (x,y) as strings of 34 bytes, as described in the following sections.

> Note: The 34-byte encoding is not injective. Each point is generally among a group of four points that share the same byte encoding.

> Note: The 34-byte encoding is not surjective. Approximately half of 34-byte strings do not encode a point (x,y).

> Note: In elliptic Diffie--Hellman (ECDH), the 34-byte encoding works well, despite being neither injective nor surjective.

4.1.1. Point encoding process

This section takes an abstract point (x,y), and outputs an encoding as a 34-byte string.

4.1.1.1. Summary

A point (x,y) is encoded by the little-endian byte representation of either x or -x, whichever fits into 34 bytes.

4.1.1.2. Details

A point (x,y) is encoded into 34 bytes, as follows.

First, ensure that x is fully reduced mod p=8^91+5, so that

0 <= x < 8^91+5.

Note: internal implementations might allow, for efficiency reasons, integer representations of x that are negative, or that are p or larger. Full reduction mod p is needed for interoperable encoding.

Second, further reduce x by a flipping its sign, as explained next. Let

x' =: min(x,p-x) mod 2^272.

Third, set the byte string b to be the little-endian encoding of the reduced integer x', by finding the unique integers b[i] such that 0<=b[i]<256 and

(x' mod 2^272) = sum (0<=i<=33, b[i]*256^i).

Pseudocode can be found in Appendix C.

Note: The loss of information that happens upon replacing x by -x corresponds to applying complex multiplication by i on the curve, because i(x,y) = (-x,iy) is also a point on the curve. (To see this: note 2(iy)^2 = -(2y^2) = -(x^3+x) = (-x)^3+(-x).) In many applications, particularly Diffie--Hellman key agreement, this loss of information is carried through to the final shared secret, which means that Alice and Bob can agree on the same secret 34 bytes.

In ECC systems where the original x-coordinate and the decoded x-coordinate need to match exactly, the 34-byte encoding is probably not usable unless the following pre-encoding procedure is practical:

Given a point x, where x is larger than min(x,p-x), first replace x by x'=p-x, on the encoder's side, using the new value x' (instead of x) for any further step in the algorithm. In other words, replace the point (x,y) by the point (x',y')=(-x,iy). Most algorithms will also require a discrete logarithm d of (x,y), meaning (x,y) = [d] G for some point G. Since (x',y') = [i](x,y), we can replace by d' such that [d']=[i][d]. Usually, [i] can be represented by an integer, say j, and we can compute d' = jd (mod ord(G)).

4.1.2. Point decoding process

4.1.2.1. Summary

To decode a 34-byte string, interpret the bytes as little-endian represented integer, which becomes the x-coordinate. When needed, public-key validation is done, to ensure that x is valid for a point on the curve. If needed, the y-coordinate is recovered from x.

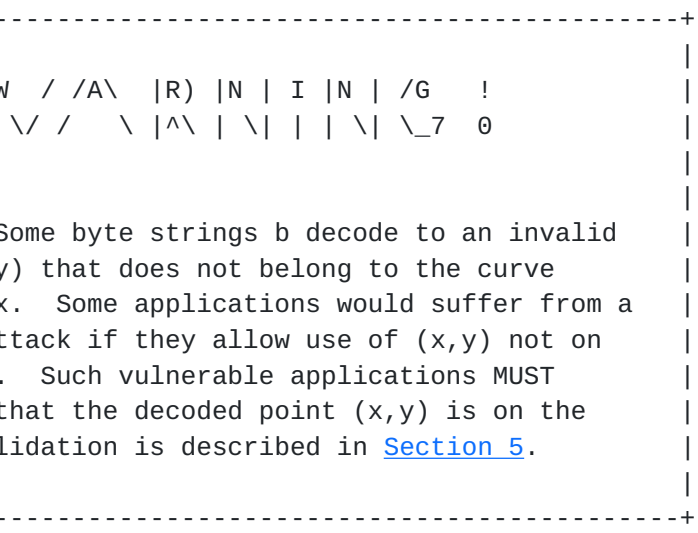
4.1.2.2. Detail

If byte i is b[i], with an integer value between 0 and 255 inclusive, then

x = sum(0<=i<=33, b[i]*256^i)

Note: a value of -x (mod p) will also be suitable, and results in a point (-x,y') which might be different from the originally encoded point. However, it will be one of the points [i](x,y) or -[i](x,y) where [i] means complex multiplication by [i].

In many cases, such as Diffie--Hellman key agreement using the Montgomery ladder, neither the original value of coordinate x (among x and -x) nor coordinate y of the point is needed. In these cases, the decoding steps can be considered completed.



In cases where a value for at least one of y, -y, iy, or -iy is needed (such as in Diffie--Hellman key agreement using Edwards coordinates), a candidate value for y can be obtained by computing a square root:

y = ((x^3+x)/2)^(1/2).

Note: Recovery of y can be combined with validation of x.

In some applications (but not Diffie--Hellman), it is important for the decoded value of x to match the original value of x exactly. In that case, the encoder should use the procedure that replaces x by p-x, and adjusts the discrete logarithm appropriately. These steps can be done by the encoder, with the decoder doing nothing.

4.2. OPTIONAL: uncompressed point representation

A speed-optimized version of elliptic curve Diffie--Hellman over curve 2y^2=x^3+x/GF(8^91+5) might be fastest if the peer's public key is supplied as an uncompressed point (x,y), because of methods like twisted Edwards coordinates.

In an arbitrary point, each coordinate uses at most 274 bits, in the standard bit representation. These 274 bits can be fit into 35 bytes. Little-endian ordering of bytes is used.

Both coordinates together fit into a 70-byte string, with the 35 bytes of x appearing first in the string.

Each point (except the identity point O, point-at-infinity) has exactly one 70-byte uncompressed representation. Each 70-byte string represents at most one point.

Converting back and forth between uncompressed and compressed representations is straightforward in principle: just decode to a point, and then re-encode with the other format. Each compressed byte string typically represents four different point ((x,y), (-x,iy), (x,-y), (-x,iy)), which have four different uncompressed representations.

Given a compressed representation, there exists a point whose uncompressed version shares the first 34 bytes with the given compressed, because the decoding of a compressed representation gives an x < 2^272, meaning x can be encoded in 34 bytes. The next byte (which is the 35th byte, which is also byte 34 in the 0-indexed array) of the uncompressed representation is set to zero. The next 35 bytes are compute from a y recovered from x. The most expensive step of computing is the square root operation, since y = ((x^3+x)/2)^(1/2).

Given an uncompressed representation (b), if the 35th byte (b[34]) is zero, then the compressed representation is just the first 34 bytes of the uncompressed representation. Otherwise, use the fist 35 bytes to compute x, then put p-x mod p into little-endian to form the compressed representation.

Point validation for uncompressed 70-byte string is more efficient than for compressed points. To check that the point is on the curve, just check that the curve equation 2y^2=x^3+x holds, which is more efficient than computing a Legendre symbol (the most expensive step of the checking that the point is on the curve given only x).

4.3. OPTIONAL: Representation of scalar multipliers

Scalars (integer point multipliers) sometimes need to be encoded as byte strings. Typical examples are the following applications.

- Digital signature in ECC generally require scalar encodings. This draft does not specify signature algorithms in detail, only providing some general suggestions.

- An ECC implementation needs to store scalars, over at least short time period, because often scalars are used at least twice, and must be stored between these two uses. For example, in elliptic curve Diffie--Hellman, Alice has scalar a, and she sends Bob point aG. Alice keeps scalar a until she receives point B back from Bob. Alice then applies a to B, computing aB. If a is ephemeral, she then deletes a. While waiting for B, Alice must store a. An implementation is free to use any encoding to store scalars. Implementation are often constructed in modular pieces. Any pieces handling the same scalar need to store the scalar in a consistent, interoperable way.

In elliptic curve Diffie--Hellman would typically using a base point G of large prime order q. The size of q is approximately p/72. The value of scalar s usually only matters mod q. An implementation can reduce s, by replacing s by s mod q. This ensures that s<q. Since q < 2^267 < 256^34, a value s can be represented in 34 bytes.

Alternatively, an implementation might instead to generate and store scalar s as a uniformly random 34-byte integer, so that 0 <= s < 256^34. Because of the approximation 256^34 ~ (36*q) (1 - 2^(-130)), scalar s chosen and represented this way has negligible bias when reduced mod q. Besides, elliptic curve Diffie--Hellman likely tolerates much higher than a digital signature would.

Finally, little-endian byte encoding of scalars is recommended, if only for a little bit of consistency the little-endian byte encoding of field elements.

4.4. OPTIONAL: Representing strings as points using Elligator i

In niche applications, it may be desired to encode an arbitrary string as a point on a curve. Example reasons to encode arbitrary 34-byte strings include:

- Encoding passwords (or their hashes) in a password-authenticated key exchange (PAKE).

- Hiding the fact that ECC is being used, by making sure that ECC public keys have the same probability distribution as uniformly random 34-byte strings.

To this end, this section sketches a method to reversibly encode an arbitrary 34-byte string as a point.

Note: To encode variable-length strings as points, one can first compute a 34-byte hash of the variable-length string, and then encode the hash. Such encoding of variable-length strings cannot be reversible.

Note: The point decoding scheme of Section 4.2 does not suffice to encode strings, because only about half of all 34-byte strings are decodable.

Note: The string-as-point encoding has the the property that only about half of all points are decodable as 34-bytes strings. Encoding a uniformly distributed 34-byte string as a point yields non-uniformly distributed points. In other words, the byte string is arbitrary, but the resulting points are not arbitrary.

The encoding is called Elligator i.

Note: The Elligator i encoding is a minor variation of the Elligator 2 construction [Elligator], introduced in [B1]. A minor variation is necessary, because Elligator 2 fails for curves with j-invariant 1728. Curve 2y^2=x^3+x has j-invariant 1728. The fix is just a minor tweak to Elligator 2, hence the bulk of the name is retained.

Fix a square root i of -1 in the field in GF(8^91+5). For example, 2^(8^89+1) mod 8^91+5.

To encode a 34-byte string b,

1. Let b represent a field element r, using little-endian base 256.

2. Compute x = i-3i/(1-ir^2). Let j=1.

3. If 2y^2=x^3+x has no solution y, then replace x by x+i and j by j+1.

4. Find two solutions y[1] and y[2] to 2y^2=x^3+x, such that y[1]<y[2].

5. Compute y=y[j].

Now (x,y) is a point on the curve 2y^2=x^3+x.

The Elligator i encoding is reversible, because it has the decoding sketched below.

The input to decoding is a point (x,y). The point (x,y) has encoded a byte string b. The job decoding is to recover b from (x,y). For a point (x,y) that does not encode any b, the decoding reports an error.

If y>p-y, replace x by x-i. Compute s = -i - 3/(i-x). Let r = sqrt(s). (If s has no square root, then report an error.) If r > p-r, replace r by p-r. Write r in little-endian base 256 to get a 34-byte string b.

Note: Just to illustrate a contrast between Elligator i encoding and the normal point encoding, consider the useless example of applying both encodings. Start with 34-byte string b. Apply Elligator i encoding to get a point (x,y). Apply the point encoding to (x,y) to get a 34-byte string b'. In summary, b'=encode(encode(b)). The byte string b' has no significant relation to b. The map b->b' from 34-byte strings to themselves is lossy (non-injective) with ratio ~4:1, and the image set is about one quarter of all 34-byte strings.

5. Point validation

In elliptic curve cryptography, scalar multiplying an invalid public key by a private key risks leaking information about the private key.

To avoid leaking information about a user private key, the user can validate that the peer's public key is a point on the curve, or even that it is point in the correct subgroup.

5.1. Schedules for point validation

This section specifies three schedules (mandatory, simplified, and minimal) for deciding when to validate whether a given point (x,y) is on the curve 2y^2=x^3+x/GF(8^91+5).

5.1.1. Mandatory schedule for point validation

As a precaution, an implementation MAY opt to apply a mandatory schedule for point validation, meaning every public key (and point) is validated, before subsequent use.

5.1.2. Simplified schedule for point validation

A small, general-purpose, implementation aiming for high speed might not be able to afford the cost of the mandatory schedule for validation from Section 5.1.1, because each validation of a 34-byte encoded point costs about 10% of a scalar multiplication.

As a practical middle ground, an implementation MAY opt to apply simplified validation, which is the rule is that a not-yet-trusted peer's public key is validated before being scalar multiplied by a static secret key.

Note: The simplified schedule for validation implies that, when the secret key is ephemeral (for example, used in one Diffie--Hellman transaction), the peer's public key need not be validated.

Note: The simplified schedule validation implies that when the point being scalar multiplied is a known valid fixed point, or a previously validated public key (including a public key from a certificate in which the certification authority has a policy to valid public keys), then validation is not needed.

5.1.3. Minimal schedule for point validation

An implementation MAY opt to use a minimal schedule for point validation, meaning doing as little point validation as possible, just enough to resist known attack against the implementation, reducing the risk to a negligible level.

The curve 2y^2=x^3+x is not twist-secure: using the Montgomery ladder for scalar multiplication is not enough to thwart invalid public key attacks.

However, consider a static hashed-ECDH implementation implemented with a Montgomery ladder, such that the static secret key is used in at most ten million times hashed-ECDH transactions. Even if exposed to invalid points on the twist, the security risk is nearly negligible. Therefore, the minimal validation would not validate the peer's public keys.

5.2. Point validation process

Upon decoding a 34-byte string into x, the next step is to compute z=2(x^3+x). Then one checks if z has a nonzero square root (in the field of size 8^91+5). If z has a nonzero square root, then the x represents a valid point, otherwise x is invalid.

Equivalently, one can check that x^3 + x has no square root (that is, x^3+x is a quadratic non-residue).

To check z for a square root, one can compute the Legendre symbol (z/p) and check that is 1. (Equivalently, one can check that ((x^3+x)/p)=-1.)

The Legendre symbol can be computed using Gauss' quadratic reciprocity law, but this requires implementing modular integer arithmetic for integral moduli smaller than 8^91+5.

Instead, one can compute the Legendre symbol using powering in the field: (z/p) = z^((p-1)/2) = z^(2^272+2). This is much slower than using quadratic reciprocity.

More generally, in signature applications (such as [B2]), where the y-coordinate is also needed, the computation of y, which involves computing a square root will generally include an implicit check that x is valid.

OPTIONAL: In some situations, it is also necessary to ensure that the point has large order, not just that it is on the curve. For points on this curve, each point has large order, unless it has torsion by 12. In other words, if [12]P != O, then the point P has large order. Most applications of elliptic curve Diffie--Hellman do not require this check.

OPTIONAL: In some situations, it may be necessary to ensure that a point P also has a prime order q = ord(G). One method to check this is to compute that [q]P, checking that [q]P = O. An alternative method is to try to solve for R in the equation [12]R=P, which involves methods such as division polynomials. The alternative has potential advantage of being faster than computing [q]P, but the clear disadvantage of requiring extra algorithms (beyond the scalar multiplication already necessary for Diffie--Hellman). Most applications of elliptic curve Diffie--Hellman do not require this check.

6. IANA considerations

This document requires no actions by IANA, yet.

7. Security considerations

No cryptographic algorithm is without risk.

Potential security risks of 2y^2=x^3+x/GF(8^91+5) are listed in this section. These risks are worth considering, before deciding to use 2y^2=x^3+x/GF(8^91+5).

This section aims to thoroughly describe the risks, perhaps erring on the side over-stating many of the risks.

Absolute risk is difficult to quantify, especially against unknown, potential attacks. Relative risk is slightly easier to qualify, if a comparable cryptographic system is available as a benchmark.

(Relative risk comparison to no cryptography is sometimes sensible. In a less exposed system, such as inter-process communication, perhaps, cryptography, especially ECC, might not provide enough benefit to justify the cost. In this case, the risk of ECC is the wasted runtime. Conversely, in over-exposed systems, all data gets made public quickly, or attacked easily, so securing data-in-transit with ECC might only provide a false sense of security.)

Most of the security risks of 2y^2=x^3+x/GF(8^91+5) listed are compared to the risks of a typical generic curve in ECC, or to the risks of specific well-established curves in ECC (such as NIST P-256 and Curve25519).

> Note: Because 2y^2=x^3+x/GF(8^91+5) MUST be used only in multi-curve ECC, comparison to other curves is mainly for selection of curves in a multi-curve ECC implementation, from a pool of curves.
>
> Note: For possible security benefits of 2y^2=x^3+x/GF(8^91+5), see Appendix A. This section and Appendix A are not entirely independent, since they discuss two sides of the same coin.

7.1. Field choice

The field size 8^91+5 has the following risks.

See [B1] for further discussion about the relative merits of 8^91+5.

7.1.1. A special prime

8^91+5 is a special prime that might lead a mathematical attack speeding up the elliptic curve discrete logarithm problem (ECDLP). Known correlations between ECC field size and ECDLP attacks are evidence of this risk. For some curve equations, the supersingularity, and thus security due to vulnerability to MOV attack, depends on the prime of the field size. For some curve equations, the curve size is related in a simple way to the field size, causing a potential correlation between the field size and the effectiveness of an attack, such as the Pohlig--Hellman attack. The curve 2y^2=x^3+x/GF(8^91+5) resists these well-known field-size-dependent ECDLP attacks. The risk is that unpublished field-size-dependent ECDLP attacks might yet exist, and might then apply to 2y^2=x^3+x/GF(8^91+5).

Other special field sizes are used by some other standard curves. The standard curves NIST P-256 and Curve25519 use special prime field sizes, with specialness quite similar to the specialness of 8^91+5. Arguably, these other standards with special field curves suffer a risk similar to that of 2y^2=x^3+x/GF(8^91+5) from unpublished field-size-dependent ECDLP attacks.

Other standard curves, such as the Brainpool curves, somewhat mitigate this risk of unpublished field-size-dependent ECDLP attacks by using pseudorandom field sizes.

7.1.2. Risk of 64-bit integer overflow

8^91+5 arithmetic implementation, while implementable in five 64-bit words, has some risk of overflowing, or of not fully reducing properly. A smaller field, such as that used in Curve25519, support simpler modular reduction and overflow-avoidance properties, if using five 64-bit words for field arithmetic.

7.1.3. Excessive size for 128-bit security

8^91+5 is well-above 256 bits in size. A user aiming to resist 2^128 step attacks, could use a smaller, and presumably faster, curve, and still have Pollard rho attacks taking 2^128 elliptic curve operations. Such a user of ECC field size 8^91+5 therefore uses unnecessary communication and computation to protect their 128-bit keys. In other words, the extra cost for exceeding 256 bits is wasteful, and arguably a form of denial of service, if the user had opportunity to devote the cost to other security protections.

7.1.4. Non-maximality among six-symbol (DEC) primes

8^91+5 is smaller than three other primes: 8^95-9, 9^99+4 and 9^87+4 that share the same decimal exponential complexity as 8^91+5: six symbols. Curves defined over larger field size have better Pollard rho security (against ECDLP attacks), but 8^91+5 has better security than these curves, against other types of attacks.

The primes 9^99+4 and 9^87+4 are not close to a power of two. This makes implementing them in software almost twice as slow as 8^91+5. The extra runtime is a weak denial-of-service attack. A more generic form of modular reduction is needed. Generic modular reduction might be more prone to implementation attacks such as side channel attacks.

The prime p'=8^95-9 is quite similar to p=8^91+5 in many aspects. Being larger than p, the prime p'=8^95-9 has a slightly greater risk than p=8^91+5 has of an arithmetic overflow implementation fault in field arithmetic. Field size 8^95-9 has more complicated powering algorithms for computing field inverses, Legendre symbols, and square roots, because the binary expansion of numbers like p'-2 have many 1-bits, unlike p-2. This discrepancy arises because p'=8^95-9 is just a little above of power of two, while p=8^91+5 is just a little below a power of two. The more complicated powering algorithms might make an implementation a little slower. Worse, they may require more complicated code, adding to the burden of debugging, and increasing the chance of implementation error.

7.1.5. Smaller five-symbol field sizes

8^91+5 is smaller than field size 2^283 (used in ECC, for curve sect283k1 [SEC2], [Zigbee]). Several other five-symbol and four-symbol field sizes (such as 9^97) are also larger than 8^91+5. When used in ECC, 8^91+5 therefore provides less Pollard rho security than such larger field sizes.

These larger field sizes are not primes, they are composite prime powers. Composite field sizes in ECC have two security risks:

- Recent progress in ECDLP attacks over composite fields [HPST] and [Nagao] is a major red flag against ECC over composite fields, suggesting that ECC over composite fields is riskier, and much closer to being broken.

- Software speed for large-characteristic field arithmetic is typically much higher than small-characteristic field arithmetic. Actually, the speed advantage is for software that runs on general-purpose hardware with dedicated 64-bit integer multiplication circuits. The composite field sizes larger than 8^91+5 have small characteristic. Software for small-characteristic field sizes is not only slower, but more complicated, and therefore more prone to implementation faults.

7.1.6. Vulnerability to faulty hardware

Efficient software implementation of 8^91+5, and other large-characteristic fields, has security that relies of hardware for 64-bit integer arithmetic.

Corrupted hardware might produce incorrect results for a single pair of inputs. Detecting such a hardware error by exhaustive search would be infeasible, because there are 2^128 possible pairs of inputs. An attacker might be able to find this hardware error by examining the hardware directly. Yet, a typical user might not be able to detect this hardware error, or to prevent it.

In this case, the attacker might be able to induce the hardware user, by supplying the user with a maliciously crafted the public key. In this case, the public key would cause the provide the faulty input values into the hardware. The resulting error might cause the user to leak their secret key to the attacker, as in [bug attacks] and [IT].

7.1.7. Other measures of complexity

Decimal exponential complexity means the minimum number of the symbols from the set

0 1 2 3 4 5 6 7 8 9 + - * ^ ( )

to specify a number, with the usual meanings of the symbols. (The symbols * and ^ are ASCII adaptations of traditional mathematical notations, and their meaning has become fairly standard in mathematical programming.)

Decimal exponential complexity is only one way to measure Kolmogorov complexity of a non-negative integer. Other measures of complexity measures allow larger primes to be expressed in six or even fewer symbols.

The factorial operation is often notated with an exclamation mark! Primes larger than 8^91+5 are then expressible in five symbols: 94!-1 is a 485-bit prime number, expressible in five symbols. Such numbers -- so far as I know -- are not close to a power of two. Therefore, they are likely have similar inefficiency and implementability defects to the primes 9^99+4 and 9^87+4, which are also far from a powers of two.

At some point, increasing the size of the symbol set should be regarded as an increase in the Kolmogorov complexity. Arguably, the exclamation mark notation favors the factorial operation over exponentiation: 5040=7! seems no simpler than 128=2^7.

It is generally understood that there can be no absolutely objective and best measure of Kolmogorov complexity.

However, in [B3], an alternative measure of Kolmogorov complexity is considered, that aims to be somewhat objective. It follows Godel's ideas for the most fundamental definition computable functions, which Turing later proved to be equivalent in computing power to his tape machines.

Unlike decimal exponential complexity, the system in [B3] has no built-in preference for decimal, or even for standard arithmetic operations like addition, multiplication, and exponentiation. Instead, all computations must be built up from the successor function and primitive recursion (or minimization recursion, but that should be avoided).

The absolute complexity measure in [B3] is difficult to determine for a given prime. Upper bounds are easy to find, by exhibiting descriptions, but it is difficult to determine the shortest description. So far, it seems that the prime 2^521-1 has lower complexity than 8^91+5, because, among the descriptions found so far, 2^521-1 currently has shorter description than the shortest yet found for
8^91+5.

The main downside of 2^521-1 compared to 8^91+5 is its larger size (almost twice the bit length), and also more complicated powering algorithms for inversion, Legendre symbols and square roots.

7.2. Curve choice

There are several security risks that might be associated with specific curve 2y^2=x^3+x/GF(8^91+5).

7.2.1. Special curve equation

The curve equation 2y^2=x^3+x is special, among elliptic curve equations. To see how special it is, we list some of its features.

- Its elliptic curve discriminant is low, at 4, whereas random curves have discriminants that average around p/2.

- Its j-invariant is 1728, whereas random curves have j-invariants that average around p/2.

- Its automorphism group has size 4, whereas random curves have automorphism group of size 2, with overwhelming probability.

- Its curve equation has three monomial terms, whereas random curves, have (all isomorphic) curve equations with at least 4 terms,
with overwhelming probability.

- It has complex multiplication by i, with [i] being a linear isogeny.

The specialness of 2y^2=x^3+x is well understood in elliptic curve theory.

Miller, in 1985, already suggested exercising prudence when considering such special curves. More precisely, Miller suggested using curve equation y^2=x^3-ax, because if a is a quadratic non-residue, then the curve has p+1 point, because it is supersingular. This saves the trouble of point-counting. But Miller predicted -- correctly! -- that this convenience is not free, it may be correlated with a risk. Indeed, two attacks were subsequently found.

The two published attacks do not impact 2y^2=x^3+x/GF(8^91+5). The risk is that similar unpublished attacks might affect 2y^2=x^3+x/GF(8^91+5).

Menezes, Okamoto and Vanstone (MOV) found an attack on elliptic curves of low embedding degree. The curve 2y^2=x^3+x/GF(8^91+5) is not vulnerable to the MOV attack, because it has high embedding degree. The concern is that 2y^2=x^3+x/GF(p') for different primes p', is vulnerable to the MOV attack. It is known to be supersingular, for about half of all primes p'.

> Note: To repeat, because the MOV attack was several years later than Miller's origin caution from 1985, Miller's prudence seems prescient. Perhaps he was also prescient about yet other potential attacks (still unpublished), and these attacks might affect 2y^2=x^3+x/GF(8^91+5). Perhaps, we continues to Miller's prudence.

Gallant, Lambert and Vanstone found ways to slightly speed up Pollard rho attacks against ECDLP, when the curve has an efficient endomorphism. This attack makes a modest speed-up for binary Koblitz curves (2^7 times speed-up), but for 2y^2=x^3+x/GF(8^91+5) makes only a minor speed-up in the ECDLP attack (2 times speed-up), not much more than the speed-up from from efficient arithmetic.

Many other standard curves, including NIST P-256 [NIST-P-256], Curve25519, Brainpool [Brainpool], do not have complex multiplication endomorphism by i, or any other low-degree endomorphism. Essentially, these curves adhere more fully to Miller's 1985 advice to be prudent about special curves.

Yet other (fairly) standard curves do, such as NIST K-283 (used in [Zigbee]) and secp256k1 (see [SEC2] and [BitCoin]). Furthermore, it is not implausible [KKM] that special curves, including those efficient endomorphisms, may survive an attack on random curves.

7.2.2. Twist insecurity

The curve 2y^2=x^3+x over 8^91+5 is not twist-secure.

An implementer might use the Montgomery ladder in Diffie--Hellman and might also re-use private keys. An implementer might imitate an implementation of a twist-secure curve like Curve25519. The twist-secure implementation would use a Montgomery ladder, and skip public-key validation.

Skipping public-key validation with a re-used secret scalar in a Montgomery ladder implementation might leak information about the re-used secret scalar, and severely weaken its security.

This document provides ample warnings, but an implementer might be too busy to read the document, or might only given a small excerpt of this document, excluding the warnings about public-key validation. An implementer might accidentally implement public-key validation with a bug that does public-key validation incorrectly.

Several other standardized curves, such as NIST P-256, are also not twist-secure. The risk from being twist-insecure for these curves to quite similar to the risk for 2y^2=x^3+x/GF(8^91+5).

Curves like NIST P-256 might have less risk from being twist-insecure, because they not Montgomery curves (and not even isomorphic to Montgomery curves). An implementer has less incentive to use a Montgomery ladder for these curves, because there is less efficiency gain compared to more conventional scalar multiplication algorithms. Conventional scalar multiplication algorithms are already well-known to require public-key validation.

Actually, the Montgomery ladder might not be the fastest scalar multiplication algorithm for 2y^2=x^3+x/GF(8^91+5). So, it is unclear how the risk of twist-insecurity really compares between it and NIST P-256.

7.2.3. Point order not near a power of two

The base point G has prime order q which is not close to a power of two.

> Note: Its leading hexadecimal expansion 71C71C... and half of leading digits (or hex-nibbles) match that of floor((8^90)/9), and q/2^266 is approximately 1.7778. See A.4.4.1. for the exact value of q.

Digital signature scheme, such as ECDSA, have a known security vulnerability, first discovered by Bleichenbacher in the case of DSA, when q is not close to a power of two. An per-message secret k, essentially an ephemeral secret scalar, is computed when generating a signature. If the probability distribution k is not uniform in [0,q-1], in the sense that some large sub-intervals are about twice as likely as others, then the signatures leak information about the static signing key d. A leak in k means a leak in d. With enough signatures with a leaky k, eventually the leaks in d are enough to reveal the whole of d. Once d is revealed, the attacker can forge arbitrary signatures.

A typical implementer generates secret scalar k by generating a uniformly random byte string b, with b will be uniformly distributed in an interval [0,2^m-1], and then setting k = b mod q. (Sometimes, the computation b mod q is delayed until the signing equation, but in all cases b mod q is the effective value for k.) If 2^m is chosen as the power of two nearest to q, then the effective k will have a bias, unless q is very close to 2^m. For 2y^2=x^3+x/GF(8^91+5), the number q is not very close to 2^m.

Signature standards mitigate this attack by several methods. Choosing longer byte string b, with max value 2^m > 2^64*q, reduces the bias of k to a negligible amount. For shorter byte strings, it is possible to re-generate b until it belongs to an an interval [0,uq-1] for some small integer u.

By contrast, many standard curves, including NIST P-256 and Curve25519, already have q close to a power of two. The reason for this is that the field size is close to a power of two, and the cofactor is also a power of two (usually 1,2,4, or 8).

7.2.4. Vulnerability to Cheon-type attacks

The Brown--Gallant--Cheon attack can find d in q^(1/3) computations using q^(1/3) queries to an oracle that compute [d]P from any given input point P.

The attack is not very serious because of large number of queries it requires. Variations of the attack work with fewer queries but use more computation. As the number of queries approaches 0, then the computation approaches q^(1/2), matching the cost of Pollard rho.

The attack is also not very serious, because only a few applications of ECC provide such an oracle to compute [d]P. Static elliptic curve Diffie--Hellman usually hashes [d]P, so does not provide such an oracle (unless the hash fails to be one-way).

The details of attack include making u queries to the oracle, where u is a factor of q-1 or q+1. The computation phase then takes about (q/u)^(1/2) steps. Putting u ~ q^(1/3), if possible, approximately minimizes the sum of the number of queries and off-line computation steps.

A general mitigation to the attack is to choose q such q-1 and q+1 have no factors u in a range that makes the attack impactful. Such a curve could be called Cheon-resistant. A Cheon-resistant curve can be used in protocols that provide the static Diffie--Hellman oracle, without worrying about degrading security from a large number of queries.

Most standardized curves, including NIST P-256, Curve25519 and the Brainpool curves, are not Cheon-resistant. Curve 2y^2=x^3+x/GF(8^91+5) is not Cheon-resistant either.

7.2.5. Small-subgroup confinement attacks

A fifth risk is a small-subgroup confinement attack, which can also leak a few bits of the private key. Curve 2y^2=x^3+x over 8^91+5 has 72 elements whose order divides 12.

7.3. Encoding choices

As in all ECC, projective coordinates are not suitable as the final representation of an elliptic curve point, for two reasons.

- Projective coordinates for a point are generally not unique: each point can be represented in projective coordinates in multiple different ways. So, projective coordinates are unsuitable for finalizing a shared secret, because the two parties computing the shared secret point may end up with different projective coordinates.

- Projective coordinates have been shown to leak information about the scalar multiplier [PSM], which could be the private key. It would be unacceptable for a public key to leak information about the private key. In digital signatures, even a few leaked bits can be fatal, over a few signatures [Bleichenbacher].

Therefore, the final computation of an elliptic curve point, after scalar multiplication, should translate the point to a unique representation, such as the affine coordinates described in this specification.

For example, when using a Montgomery ladder, scalar multiplication yields a representation (X:Z) of the point in projective coordinates. Its x-coordinate is then x=X/Z, which can be computed by computing the 1/Z and then multiplying by X.

The safest, most prudent way to compute 1/Z is to use a side-channel resistant method, in particular at least, a constant-time method. This reduces the risk of leaking information about Z, which might in turn leak information about X or the scalar multiplier.

Fermat inversion, computation of Z^(p-2) mod p, is one method to compute the inverse in constant time (if the inverse exists).

7.4. General subversion concerns

The main motivation of curve 2y^2=x^3+x over 8^91+5 is to minimize the risk of curve subversion via a backdoor ([Gordon], [YY], [Teske]).

A security skeptic ought to consider any security technology's appearance in a standards development organization document with suspicion, as an possible effort at subversion. (See [BCCHLV] for some detailed analysis.) This document, despite its intended experimental status, can be seen as possible subversion. Other standards for curves could be also seen as even more successful efforts a subversion.

> Note: A skeptic needs to be careful not to become paranoid. Interoperable ECC has a strong need for standardized curves. If there is a non-subverted secure curve, standardizing it is good for security.

A skeptic needing to choose between standardized curves can then objectively examine the curve's intrinsic merits both or, failing that, perhaps objectively examine reputation of the (alleged) author of the standard, making an ad hominem argument.

This document asks all users, including skeptics, to prefer mainstream curves like NIST P-256 or Curve25519, to this document's curve 2y^2=x^3+x/GF(8^91+5). This document invites users to use two or more curves in ECC, with 2y^2=x^3+x/GF(8^91+5) as a second line of defense.

The reader is encouraged to take a skeptical viewpoint of curve 2y^2=x^3+x/GF(8^91+5), and of other curves. Skeptical users of the curve are expected to make a rational as possible decision to add 2y^2=x^3+x/GF(8^91+5) to the list of curves used in multi-curve ECC.

To paraphrase, users seriously worried about subverted curves (or other cryptographic algorithms), either because they estimate as high either the probability of subversion or the value of the data needing protection, have good reason to like 2y^2=x^3+x/GF(8^91+5) for its compact description.

Nevertheless, the best way to resist subversion of cryptographic algorithms seems to be combine multiple dissimilar cryptographic algorithms, in a strongest-link manner. Diversity hedges against subversion, and should the first defense against it.

7.5. Risk of low age and eyes (aegis)

The exact curve 2y^2=x^3+x/GF(8^91+5) was (seemingly) first described to the public in 2017 [AB]. So, it has a very low age, at least compare to more established curves, like NIST P-256 (from 1999) and Curve25519 (from 2005).

The curve 2y^2=x^3+x/GF(8^91+5) it has not been submitted for a publication to any formally peer-reviewed academic cryptographer forum, such as the IACR conferences like Crypto and Eurocrypt. It has most like been reviewed by very few eyes.

Arguably, other reviewers have little incentive to study it critically, for a couple reasons:

- The looming threat of a quantum computer has diverted many researchers towards studying post-quantum cryptography, such as supersingular isogeny Diffie--Hellman.

- Past CFRG disputes over NIST P-256 and Curve25519 (and other ECC alternatives) have perhaps tired some reviewers, many of whom would to prefer to concentrate on deployment of ECC, seeing disputes as delay.

The simplistic metric of aegis, meaning in age times eyes (times incentive), 2y^2=x^3+x/GF(8^91+5), scores low. Counting myself (but not quantifying incentive) it gets an aegis score of 0.1 (using a rating 0.1 of my eyes factor in the aegis score: I have not discovered any major ECC attacks of my own.) This is far smaller than my estimates (see below) some more well-studied curves.

Nonetheless, the curve 2y^2=x^3+x/GF(8^91+5) has some similarities to some of the better-studied curves with much higher aegis:

- Curve25519: has field size 8^85-19, which a little similar to 8^91+5; has equation of the form by^2=x^3+ax+x, with b and a small, which is similar to 2y^2=x^3+x. Curve25519 has been around for over 10 years, has (presumably) many eyes looking at it, and has been deployed thereby creating an incentive to study. An estimated aegis for Curve25519 is 10000.

- NIST P-256: has a special field size, and maybe an estimated aegis of 200000. (It is a high-incentive target. Also, it has received much criticism, showing some intent of cryptanalysis. Indeed, there has been incremental progress in finding minor weakness (implementation security flaws), suggestive of actual cryptanalytic effort.) The similarity to 2y^2=x^3+x over 8^91+5 is very minor, so very little of the P-256 aegis would be relevant to this document.

- secp256k1: has a special field size, though not quite as special as 8^91+5, and has special field equation with an efficient endomorphism by a low-norm complex algebraic integer, quite similar to 2y^2=x^3+x. It is about 17 years old, and though not studied much in academic work, its deployment in Bitcoin has at least created an incentive to attack it. An estimated aegis for secp256k1 is 10000.

- Miller's curve: Miller's 1985 paper introducing ECC suggested, among other choices, a curve equation y^2=x^3-ax, where a is a quadratic non-residue. Curve 2y^2=x^3+x is isomorphic to y^2=x^3-x, essentially one of Miller's curves, except that a=1 is a quadratic residue. Miller's curve may not have been studied intensely as other curves, but its age matches that ECC itself. Miller also hinted that it was not prudent to use a special curve y^2=x^3-ax: such a comment may have encouraged some cryptanalysts, but discouraged cryptographers, perhaps balancing out the effect on the eyes factor the aegis. An estimated aegis for Miller's curves is 300.

Some of the direct estimates of aegis for the similar curves could be counted towards to 2y^2=x^3+x/GF(8^91+5), as an indirect form of aegis. Some obvious cautions to the reader:

- Small changes in a cryptographic algorithm can cause large differences in security. Security arguments based on similarity in cryptographic schemes should be given low priority.

- Security flaws have sometimes remained undiscovered for years, despite both incentives and peer reviews (and lack of hard evidence of conspiracy). The eyes factor of the aegis score is very subjective. It is vulnerable to false positives via a herd effect. Despite this caveat, it is not sensible to completely ignore the eyes factor in the aegis score. Otherwise, one might deploy for cryptographic schemes that might never have been studied, which might include many insecure schemes.

7.6. Risk of ECC and multi-curve ECC

This document argues for continued use of ECC, and also for multi-curve ECC.

There is large consensus to use ECC for the time-being, but not yet much consensus for multi-curve ECC.

7.6.1. Risk of ECC, overall

Some critics do not consider ECC safe to use, even the CFRG curves like Curve25519. Two criticisms are discussed below.

7.6.1.1. Risk of hidden pre-quantum attacks on ECC

Extreme skeptics might consider all (or most) ECC to be insecure, vulnerable to some unpublished attack, runnable today using conventional pre-quantum computers.

These skeptics are outliers, given how much ECC is used today.

Nonetheless, they have arguments with some merit. They could argue that ECC is too sophisticated, in that only a few elite can claim to have truly contributed to its security analysis. Indeed, ECC is so sophisticated, that nobody can truly claim to fully understand it, especially its security.

These skeptics might prefer RSA or finite-field Diffie--Hellman.

These skeptics ought to be willing to use a hybrid of ECC in a combination with some other form of public-key cryptography.

7.6.1.2. Risk of quantum computer attacks on forward secrecy

Some users believe that the two probabilities below are so high, that ECC can no longer provide much security (in terms of secrecy).

- The probability that an attacker records their ECC-protected ciphertext.

- The probability that an attacker has, or will soon have, a quantum computer capable of breaking ECC.

These users are optimistic about quantum computers. These quantum-optimists might think that we should stop bother to ECC, maybe even right now, and take our chances with some post-quantum cryptography.

The general consensus seems to be address this quantum threat by using ECC combined with a post-quantum cryptography.

7.6.2. Multi-curve ECC might be wasteful

Milder skeptics might argue that multi-curve ECC does not add enough benefit over single-curve ECC to justify its cost.

These skeptics might infer the risk of curve-specific attacks is already low enough. Major current curves, such as NIST P-256, Curve25519 and Brainpool, already include mitigations against subversive curve-specific attacks. The existing mitigation might be enough for milder skeptics. After all, the latest curve-specific ECDLP attacks are from 2000, at least for prime-field curves, which might convince milder skeptics that curve-specific ECDLP attacks have been exhausted for prime-field curves. The skeptics might also feel that curves like Curve25519 have resolved the risk of implementation attacks to the lowest possible for ECC.

8. References

8.1. Normative References

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8.2. Informative References

To be completed.

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[B1] D. Brown. ECC mod 8^91+5. IACR ePrint, 2018. <https://ia.cr/2018/121>

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[B4] D. Brown. GLV+HWCD for 2y^2=x^3+x/GF(8^91+5). IACR ePrint, 2021. <http://ia.cr./2021/383>

[KKM] A. Koblitz, N. Koblitz and A. Menezes. Elliptic Curve Cryptography: The Serpentine Course of a Paradigm Shift, IACR ePrint, 2008. <https://ia.cr/2008/390>

[BCCHLV] D. Bernstein, T. Chou, C. Chuengsatiansup, A. Hulsing, T. Lange, R. Niederhagen and C. van Vredendaal. How to manipulate curve standards: a white paper for the black hat, IACR ePrint, 2014. <https://ia.cr/2014/571>

[Elligator] (((To do:))) fill in this reference.

[NIST-P-256] (((To do:))) NIST recommended 15 elliptic curves for cryptography, the most popular of which is P-256.

[Zigbee] (((To do:))) Zigbee allows the use of a small-characteristic special curve, which was also recommended by NIST, called K-283, and also known as sect283k1. These types of curves were introduced by Koblitz. These types of curves were not recommended by NSA in Suite B.

[Brainpool] (((To do:))) the Brainpool consortium (???) recommended some elliptic curves in which both the field size and the curve equation were derived pseudorandomly from a nothing-up-my-sleeve number.

[SEC2] Standards for Efficient Cryptography. SEC 2: Recommended Elliptic Curve Domain Parameters, version 2.0, 2010. <http://www.secg.org/sec2-v2.pdf>

[IT] T. Izu and T. Takagi. Exceptional procedure attack on elliptic curve cryptosystems, Public key cryptography -- PKC 2003, Lecture Notes in Computer Science, Springer, pp. 224--239, 2003.

[PSM] (((To do:))) Pointcheval, Smart, Malone-Lee. Projective coordinates leak.

[BitCoin] (((To do:))) BitCoin uses curve secp256k1, which has an efficient endomorphism.

[Bleichenbacher] To do: Bleichenbacher showed how to attack DSA using a bias in the per-message secrets.

[Gordon] (((To do:))) Gordon showed how to embed a trapdoor in DSA parameters.

[HPST] Y. Huang, C. Petit, N. Shinohara and T. Takagi. On Generalized First Fall Degree Assumptions, IACR ePrint 2015. <https://ia.cr/2015/358>

[Nagao] K. Nagao. Equations System coming from Weil descent and subexponential attack for algebraic curve cryptosystem, IACR ePrint, 2015. <http://ia.cr/2013/549>

[Teske] E. Teske. An Elliptic Curve Trapdoor System, IACR ePrint, 2003. <http://ia.cr/2003/058>

[YY] (((To do:))) Yung and Young, generalized Gordon's ideas into Secretly-embedded trapdoor ... also known as a backdoor.

Appendix A. Why 2y^2=x^3+x/GF(8^91+5)?

This section says why curve 2y^2=x^3+x/GF(8^91+5) can improve ECC, if used properly in multi-curve ECC.

Note: Other sections (especially 4, 5, 6, B, C, and D) cover some relatively routine ECC details about how to use 2y^2=x^3+x/GF(8^91+5). Section 8 covers some reasons why one might not want to use 2y^2=x^3+x/GF(8^91+5).

A.1. Not for single-curve ECC

Curve 2y^2=x^3+x/GF(8^91+5) is riskier than other IETF-approved curves, such as NIST P-256 and Curve25519, for at least the following reasons:

- it is newer, so riskier, all else equal, and

- it is special, with complex multiplication by i: consensus continues to agree with Miller's original 1985 opinion that using (such) special curves is not "prudent".

Koblitz, Koblitz and Menezes [KKM] somewhat dissent from the consensus against special curves. They list several plausible cases of special curves -- including some with complex multiplication -- that they argue might well be safer than random curves. (Others go even further, dismissing prudence against special curves as myth [ref-tba].)

Despite this dissent, this report adheres to the consensus, which is to prefer other curves for single-curve ECC.

The relative newness of 2y^2=x^3+x/GF(8^91+5) is not entire. The curve equation is isomorphic to one proposed by Miller in 1985, making it older than the isomorphism class of curve equations in NIST P-256 or Curve25519. The field size, the prime 8^91+5=2^273+5, is a prime likely to have been considered before the field size primes NIST P-256 or Curve25519, but probably not in an application to ECC (i.e. probably in surveys of special primes).

A.2. Risks of new curve-specific attacks

A risk for all ECC is new curve-specific attacks, especially attacks on the elliptic curve discrete logarithm problem. A new curve-specific attack could break any ECC using the affected curves.

The main benefit to ECC of curve 2y^2=x^3+x/GF(8^91+5) is to reduce this risk in multi-curve variant of ECC.

Note: an arguably larger risk, a quantum computer capable of running Shor's algorithm, looms over all of ECC. The probability of this risk is basically independent of the probability new curve-specific attack, but the impacts are heavily dependent, if a quantum attack impacts ECC, then the new curve-specific attacks are totally moot. Also, even if no quantum attack on ECC emerges, but PQC supplements or replaces ECC, then a new curve-specific attack becomes much more tolerable. For sake of argument, suppose probabilities 1% for a new curve-specific attack by 2030, and 10% for a quantum-attack on ECC by 2030. Addressing the 10% probability risk is more urgent, but there is still a 90% chance that of no-quantum-attack. Assuming that PQC is combined with ECC (instead of replacing it) and assuming that the 10% and 1% probabilities above are formally independent, then there is 0.9% probability that new-curve specific on ECC by 2030 would affect PQC+ECC systems, reducing their security to that of PQC only.

A.2.1. What would be considered a "new curve-specific" attack?

The idea of new curve-specific attacks is now discussed. The purpose is to remind the reader of the risks, by comparison to past curve-specific attacks, so that a user can estimate the benefits of addressing the risk. Ultimately, the reader should make an informed as possible decision whether the extra cost of multi-curve is warranted.

A.2.2.1. What would be considered a "new" attack?

The "new" in "new curve-specific attack" means hypothetical and not yet published, and hence, either future or hidden. This contemplates an adversary with superior cryptanalytic capability than current state-of-the-art knowledge.

A.2.2.2. What is, would be, considered a "curve-specific attack"?

The "curve-specific" in "new curve-specific attack" means that the following conditions on the attack are true

- it affects almost ECC algorithms using the specific curve (typically, if the discrete logarithm problem is easy for that curve, or in some cases, the decision Diffie--Hellman problem),

- it does not affect ECC using at least one other curve (typically, many other curves), and

- it would not affect a generic group of the same size of the secure ECC group.

Note: For example, the naive Pollard-rho attack is not "curve-specific" because it fails the second condition and third condition (it affects all curves and all generic groups of equal or smaller size than the attacked curve). The Pohlig--Hellman attack (on smooth order groups) is not curve-specific because it fails the third condition.

Note: A side-channel attack on an ECC implementation is not necessarily "curve-specific" in the strict sense above, if another ECC implementation using the same curve resists the attack. Some curves may be more prone than others to side-channel attacks, here we refer to that situation "curve-specific implementation-vulnerability".

Prime-field curves were affected by two curve-specific attacks (on the discrete logarithm): the MOV attacks, and the SASS attack, both from before 2001. For the decision Diffie--Hellman problem, a generalization of the MOV attack can be considered as curve-specific.

For non-prime-field curves, more recent curve-specific attacks have been discovered, some asymptotically polynomial-time. (To be completed.)

A.2.2.3. Rarity of published curve-specific attacks

To be completed.

The known curve-specific attacks against prime-field curves are rare in the sense of having negligible probability of affecting a random curve (over a given prime-field).

Some of these are attacks are also field-specific too.
These attacks somewhat rare among all possible non-prime-field curves (though in some cases the probability among certain class of curves is non-negligible).

If the rarity of the known curve-specific attacks carries over to any new curve-specific attacks, then truly random curves should resist the new curve-specific attacks, except with negligible probability. Honestly generated, non-random curves should also resist the new curve-specific attacks, except in the unfortunate case the new curve-specific attack is correlated with the honest curve generation criteria.

A.2.2.4. Correlation of curve-specific efficiency and attacks

To be completed.

Many of the known curve-specific attacks affected previously proposed curves, and presumably honestly generated curves. For example, supersingular curves were proposed for their slightly greater efficiency over ordinary curves, but then turned out to be vulnerable to the MOV attack. (Similarly, curves vulnerable to the SASS attack were proposed for slight efficiencies, before the SASS attack was published.) So, such correlations are not only plausible, but the real-world pattern for ECC. Accidents have already happened for such non-random curves.

Worse yet, if a non-random curve is chosen maliciously, a correlation between a hidden curve-specific attack and some sensible curve generation criteria might well make it possible for a maliciously chosen non-random curve to be made vulnerable to a hidden curve-specific attack.

A.3. Mitigations against new curve-specific attacks

Because the risk of new curve-specific attack is nonzero, applying mitigations against the risk potentially improves security, albeit at some cost.

A.3.1. Fixed curve mitigations

Often, a single fixed curve is used across a system of ECC users, generally for reasons of efficiency. This exposes the system to the nonzero risk of new curve-specific attacks.

A.3.1.2. Existing fixed-curve mitigations

Some of the better established fixed curve have sensibly included mitigations against the nonzero risk of new curve-specific attacks.

- NIST curve P-256 has coefficients derived from the output of SHA-1, perhaps aiming to avoid any new curve-specific weakness that would apply rarely to random curves, although inadequately so, because the seed input to the hash is utterly inexplicable, and plausibly manipulable.

- Bernstein's Curve25519 results from a "rigid", non-random design process, favoring efficiency over all else, perhaps eliminating intentional subversion towards a new curve-specific weakness.

- Brainpool's curves are derived using hash functions applied to nothing-up-my-sleeve numbers, perhaps aiming to mitigate both intentional subversion and accidental rare weakness.

  Note: A reasonable inference from these curves is that risk of new curve-specific attacks warranted the mitigations used (as listed above). The risk may be less now that further time has passed, because no other curve-specific attacks against prime-field curves arose in the interim. The risk is still not zero, so the mitigations may still be warranted.

A.3.1.2. Mitigations used by 2y^2=x^3+x/GF(8^91+5)

The curve 2y^2=x^3+x/GF(8^91+5) includes similar fixed-curve mitigations against the risk of new curve-specific attacks:

- a short description (low Kolmogorov complexity), aiming to have little wiggle for an intentional embedded weakness (somewhat like a nothing-up-my-sleeve number used in the Brainpool curves),

- a set of special efficiencies, such as a curve endomorphism, Montgomery form, and fast field operation (somewhat like the "rigid" properties of Curve25519 favor efficiency as a mitigation to fight off intentional embedded weakness),

- a prime field, to stay clear of recent curve-specific attacks on non-prime-field ECC.

These mitigations do not suffice to justify its use in single-curve ECC (instead of more established non-special curves).

Note: The mitigations above, like those of NIST P-256 and Curve25519, have a cost which consists mostly of a one-time computation. The mitigations are somewhat warranted, even if multi-curve ECC, because the aim of multi-curve is to hedge the risk of curve-specific attacks, so it makes sense for each individual curve to include mitigations against this risk.

A.3.2. Multi-curve ECC

This section further motivates the value of multi-curve ECC over single-curve ECC, but does specify a detailed way to do multi-curve ECC.

Multi-curve ECC is only really effective if used with a diverse set of curves. Multi-curve ECC SHOULD use a set of curves including the three curves:

NIST P-256, Curve25519, and 2y^2=x^3+x/GF(8^91+5).

Multi-curve ECC aims to further mitigate the risk of curve-specific attack, by securely combining a diverse set of curves. The aim is that at least one of the curves used in multi-curve ECC resists a new curve-specific attack (if a new attack ever appears). This aim is only plausible if the set of curves used is diverse, in features or in authorship.

This curve contributes to the diversity necessary for multi-curve ECC, with special technical features distinct from established curves NIST P-256 and Curve25519 (and Brainpool):

- complex multiplication by i (low discriminant, rather than high),

- a greater emphasis on low Kolmogorov descriptional complexity (rather than hashed coefficient or efficiency).

A.3.2.1. Multi-curve ECC is a redundancy strategy

Multi-curve ECC is an instance of a strategy often called redundancy, applied to ECC. Redundancy is quite general in that it can be applied to other types of cryptography, to other types of information security, and even to safety systems. Other names for redundant strategies include:

strongest-link, defense-in-depth, hybrid, hedged, composite, fail-safe, diversified, resilient, belt-and-suspenders, fault tolerant, robust, multi-layer, robustness, compound, combination, etc.

A.3.2.2. Whether to use multi-ECC

Multi-curve ECC mitigates the risk of new curve-specific attacks, so ought to be used instead of single-curve ECC if affordable, such as when

- the privacy of the data being protected has higher value than the extra cost of multi-curve ECC, which may be the case for at least financial, medical, or personally-identifying data, and

- ECC is only a tiny portion of the overall system costs, which would be the case if the data is human-generated or high-volume, or if ECC is combined with slow or large post-quantum cryptography (PQC).

A.3.2.2.1. Benefits of multi-curve ECC

The benefit of multi-curve ECC is difficult to quantify. The aimed benefit over single-curve ECC is extra security, in the event of a significant curve-specific attack.

No extra security results if all the curves used are the same. The curves must be diverse, so that a potential attack on one is somehow unlikely to affect the other. This diversity is difficult to assess. Intuitively, a geometric metaphor of a polygon for the space of all choices might help. Maximally distant points in a polygon tend to be vertices, the extremities of the polygon. Translating this intuition suggests choosing curves at the extremes of features.

> Note: By contrast, in a single-curve ECC, the geometric metaphor suggests a central internal point, on the grounds that each vertex is more likely to be affected to a special attack. Carrying this over to multi-curve suggests that a diverse set ought to include a non-extreme curve too.

As always, the benefit of security is really the negative of the cost of an attack, including the risk.

The contextual benefit of multi-curve ECC therefore depends very much on the application, involving the assessing both the probability of attack, and the impact of the attack.

Higher value private data has greater impact if attacked, and perhaps also higher probability, if the adversary is more motivated to attack it.

Low probability of attacks are mostly inferred through failed but extensive cryptanalysis efforts. Normally, this is only intuited, but approaches to quantifiably estimate these probabilities is possible too, under sufficiently strong assumptions.

To be completed.

A.3.2.2.2. Costs of multi-curve ECC

The cost of multi-curve ECC is fairly easy to quantify (easier than quantifying the benefit).

The cost of multi-curve is meant to be compared to the cost of single-curve ECC.

The cost ratio is approximately the number of curves used. The cost difference depends on the devices implementing the ECC.

For example, on a current personal computer, the extra cost per ECC transaction can include up to 1 millisecond of runtime and sending an extra 30 bytes or more. In low-end devices, the time may be higher due to slower processors.

The contextual cost of ECC depends on the application context. In some applications, such as personal messages between two users, the cost (milliseconds and a few hundred bytes) is affordable relative to the time users spent writing and reading the messages. In other applications, such as automated inter-device communication with frequent brief messages, single-curve ECC may already be a bottleneck, costing most of the run-time.

A.3.2.3. Applying multi-curve ECC

For key establishment, NIST recently proposed (in a draft amendment to Special Publication 800-133 on key derivation) a mechanism to support deriving a single symmetric key from the result of multiple key establishments. In summary, the mechanism is that the raw ECDH shared secrets would be concatenated and fed into a hash-based key derivation function.

An alternative would be to XOR multiple shared symmetric-key together.

So, multi-curve elliptic curve Diffie--Hellman (ECDH) key agreement could use one of these mechanism to derive a single key from multi-curve ECDH.

A mechanism to support sending more than one ECDH public key (usually ephemeral), with an indication of the curve for each ECDH key, would also be needed.

For signatures, the simplest approach is to attach multiple signatures to each message. (For signatures providing message recovery, then an approach is to apply the results, with outer signatures recover the inner signed message, and so on.)

A.4. General features of curve 2y^2=x^3+x/GF(8^91+5)

This subsection describes some general features of the curve

2y^2=x^3+x/GF(8^91+5),

presuming a familiarity with elliptic curve cryptography (ECC).

Each of a set of well-established features, such as Pollard rho security or Montgomery form, for ECC in general are evaluated and summarized for the specific curve 2y^2=x^3+x/GF(8^91+5).

Note: Interoperable ECC requires a few more details than are deducible from mathematical description 2y^2=x^3+x/GF(8^91+5) of the curve, such encoding points as byte strings. These details are discussed in Sections 4, 5, and 6.

A.4.1. Field features

The curve's field of definition, GF(8^91+5), is a finite field, as is always the case in ECC. (Finite fields are Galois field, and the field of size is p is written as GF(p).)

The field size is the prime p=8^91+5. (See the appendix for a Pratt primality certificate.)

In hexadecimal (base 16, big-endian) notation, the number 8^91+5 is

2000000000000000000000000000000000000000000000000000000000000005

with with 67 zeros between 2 and 5.

The most recent known curve-specific attacks on prime-field ECC are from 2000.

Prime fields in ECC tend be more efficient in software than in hardware.

The prime p is very close to a power of two. Primes very close to a power of two are sometimes known as Crandall primes. Reduction modulo p is more efficient for Crandall primes than for most other primes (or at least random primes). Perhaps Crandall primes are more resistant to side-channel attacks or implementation faults than than most other primes.

The fact that p is slightly larger than a power of two -- rather than slightly lower -- means that powering algorithms to compute inverses, Legendre symbols, and square roots are simpler and slightly more efficient (than would be for prime below a 2-power).

A.4.3. Equation features

The curve equation 2y^2=x^3+x has Montgomery form,

by^2=x^3+ax^2+x,

with (a,b) = (0,2). This permits the Montgomery ladder scalar point multiplication algorithm to be used, which makes it relatively efficient, and also easier to protect against side channels.

The curve 2y^2=x^3+x has complex multiplication by i, meaning an endomorphism

(x,y) -> (-x,iy).

Note: Strictly speaking, over some fields, the curve the curve has quaternionic multiplication (where it is said to be supersingular), in which case, the term "complex multiplication" is not the best fit.

The endomorphism permits the Gallant--Lambert--Vanstone (GLV) scalar multiplication algorithm, which makes it relatively efficient. See [B4] for a discussion of combining GLV with Hisil--Wong--Carter--Dawson arithmetic for twisted Edwards coordinates, as it applies to 2y^2=x^3+x/GF(8^91+5).

The GLV method can also be combined with Bernstein's two-dimensional variant of the Montgomery ladder algorithm. See [B1] for some discussion.

The curve has j-invariant 1728, because it has complex multiplication by i.

Note: The j-invariants 0 and 1728 are special in that the curves with these j-invariants have more than two automorphisms. (Relatedly, over complex numbers, the moduli space of elliptic curves is an orbifold, with exactly two non-smooth points, at j=0 and j=1728.)

A.4.4. Finite curve features

This section describes features of 2y^2=x^3+x/GF(8^91+5) as a finite curve consisting, the points (x,y) for x,y in GF(p), and also the point at infinity. In other words, these features are specific to the combination of both the finite field and the curve equation.

> Note: In algebraic geometry, these points are said to rational over k=GF(p), and the set of rational points written as E[k] = (2y^2=x^3+x)[GF(8^91+5)], to distinguish from points with coordinates in the algebraic closure of k=GF(p).

Many security properties, and a few performance properties, of ECC are specific to a finite curve.

A.4.4.1. Curve size and cofactor

The curve (of points rational over GF(8^91+5)) has size (order) 72q for a large prime q, which is, in hexadecimal,

71C71C71C71C71C71C71C71C71C71C7A4ACED12AE9418569B932B8A7B80438A9

> NOTE: Appendix E has a Pratt primality certificate for q.

So, the curve has cofactor 72.

The curve size can verified by implementing the curve's elliptic curve arithmetic, and scalar multiplying random points on the curve by the claimed size.

The size can also be partially verified with the complex multiplication theory and a little big integer arithmetic, as reviewed below.

The prime p=8^91+5 has p=1 mod 4, so a theorem of Fermat says there exist integers u and v such that p=u^2+v^2. Numbers u and v can found using a special case of Cornacchia's algorithm, and are listed further below.

Complex multiplication theory says that a curve with complex multiplication by i has size s=(u+1)^2+v^2 = p+2u+1. By negation of u, and by swapping u and v, there are four possible sizes, p+2u+1, p-2u+1, p+2v+1, p-2v+1. These are known as the (quadratic) twist sizes.

Curve 2y^2=x^3+x/GF(8^91+5) has one of these four sizes. In this case, its size s is divisible by 72, and has large prime factor q = s / 72.

The following 'bc' program includes values for u and v applicable to 2y^2=x^3+x/GF(8^91+5), verifies these calculations, and outputs q.

```
p = 8^91+5
u = 1043033027901133467787029269772887051447 69
v = 655585368017578752283604058587318062815 06
if ( p != u^2+v^2 ) { "u and v incorrect" ; halt }
s = (u+1)^2 + v^2
if ( 0 != (s % 72)) { "size not divisible by 72" ; halt}
q = s/72
q
```

Note: Theory only indicates that s has one of four values, so an extra step is needed to verify which of the four values is the size. Scalar multiplication by s is a general method. A faster method, which happens to work for 2y^2=x^3+x/GF(8^91+5), is to show that only one of the four candidate sizes is divisible by 3, and then demonstrate a point of order 3 on this curve. Symbolic calculation with elliptic curve arithmetic show that the point (x,y) has order 3 if 3x^4 + 1 = 0 in GF(p). The big integer calculation (-(1+2p)/3)^((p-1)/4) = 1 mod p shows that such an x exists in GF(p).

Note: The Schoof--Elkies--Atkin (SEA) point-counting algorithm can compute the size of any general curve, but is slower than methods for some special curves, which is why Miller had already suggested in 1985 to use special curves with equation y^2=x^3-ax. Miller suggested using supersingular curves, but restriction of the coefficient a, but those curves are insecure. Instead, we have chosen a different a (in this case a=-1 is isomorphic to 2y^2=x^3+x), which is not supersingular, but still has much easier point counting than the generic SEA algorithm.

A.4.4.2. Pollard rho security

The prime q is 267-bit number. The Pollard rho algorithm for discrete logarithm to the base G (or any order q point) takes (proportional to) sqrt(q) ~ 2^133 elliptic curve operations. The curve provides at least 2^128 security against Pollard rho attacks, with about 5 bits to spare.

Note: Arguably, the fact ECC operations are slower than symmetric-key operations (such as hashing or block ciphers), means that ECC security should be granted a few extra bits, perhaps 5-10 bits, of security when trying to match ECC security with symmetric-key security. In this case, one might say that 2y^2=x^3+x/GF(8^91+5) resists Pollard-rho with 2^140 security, providing 12 bits of extra security. The extra security can be viewed as a safety margin for error, or as an excessive to the extent the smaller, and faster curves would more than suffice to match 2^128 security of SHA-256 and AES-128.

Gallant, Lambert, Vanstone, show how to speed up Pollard rho algorithms when the group has an extra endomorphism, which would apply to 2y^2=x^3+x. The speed-up here amounts to a couple of bits in the security,

A.4.4.3. Pohlig--Hellman security

The small cofactor means the curve effectively resists Pohlig--Hellman attack (a generic algorithm to solve discrete logarithms in any group in time sqrt(m) where m is the largest prime factor of the group size).

Note: Consensus in ECC is to recommend a small factor, such as 1, 2, 4, or 8, despite the fact that, for random curves, the typical cofactor is approximately p^(1/3), which is much larger. The small cofactor helps resists Pohlig--Hellman without increasing the field size. (A larger field size would be less efficient.)

A.4.4.2. Menezes--Okamoto--Vanstone security

The curve has a large embedding degree. More precisely, the curve size 72q has q with embedding degree (q-1)/2.

This means that the discrete logarithms to base G (a point of order q) resist Menezes--Okamoto--Vanstone attack.

The large embedding degree also means that that no feasible pairings exist that could be used solve the decision Diffie--Hellman problem (for points of order q). Similarly, the larger embedding degree also means, it cannot be used for pairing-based cryptography (and it would already too small to be used for pairing-based cryptography).

Note: Intuitively, a near-miss or a close-call could describe this curve's resistance to the MOV attack. For about half of all primes P, then curve 2y^2=x^3+x is supersingular over GF(P), with embedding degree 2, making them vulnerable to the MOV attack reduces the elliptic curve discrete logarithm to the finite field discrete logarithm over GF(P^2). Miller suggested in 1985 to use isomorphic equations, y^2=x^3-ax, without knowing about the 1992 MOV attack. These special curves would then be vulnerable with ~50% chance of being, depending on the prime P. This curve was chosen in full knowledge of the MOV attack.

Note: The near-miss or close-call intuition is misleading, because many cryptographic algorithms become insecure based on the slightest adjustment to the algorithm.

Note: The non-supersingularity means that the endomorphism ring is commutative. For this curve the endomorphism ring is isomorphic to the ring Z[i] of Gaussian integers.

A.4.4.3. Semaev--Araki--Satoh--Smart security

The fact that the curve size 72q does not equal p, means that the curve resists the Semaev--Araki--Satoh--Smart attack.

A.4.4.4. Edwards and Hessian form

The cofactor 72 is divisible by 4, so the curve isomorphic to a curve with an Edwards equation, permitting implementation even more efficient than the Montgomery ladder.

The Edwards form makes possible the Gallant--Lambert--Vanstone method that used the efficient endomorphism.

The cofactor 72 is also divisible by 3, so the curve is isomorphic to a curve with a Hessian equation, which is another type of equation permitting efficient implementation.

Note: It is probably too optimistic and speculative to hope that future research will show how to take advantage by combining the efficiencies of Edwards and Hessian curve equations.

A.4.4.5. Bleichenbacher security

Bleichenbacher's attack against faulty implementations discrete-log-based signatures fully affects 2y^2=x^3+x/GF(8^91+5), because the base point order q is not particularly close to a power of two. (Some other curves, such as NIST P-256 and Curve25519, have the base point order is close to a power of two, which provides built-in resistant to Bleichenbacher's faulty signature attack.)

Note: Bleichenbacher's attack exploits the signature implementation fault of naively reducing uniformly random bit strings modulo q, the order of the base point, which results in a number biased towards the lower end of the interval [0,q-1].

So, q-uniformization of the pre-message secret numbers is critical for signature applications of 2y^2=x^3+x/GF(8^91+5). Various uniformization methods are known, such as reducing extra large numbers, repeated sampling, and so on.

A.4.4.6. Bernstein's "twist" security

Unlike Curve25519, curve 2y^2=x^3+x/GF(8^91+5) is not "twist-secure", so a Montgomery ladder implementation for static private keys often requires public-key validation, which is achievable by computation of a Legendre symbol related to the received public key.

In particular, a Montgomery ladder x-only implementation that does not implement public-key validation will process a value x for which no y satisfying the equation exists in GF(p). More precisely, a y does exist, but it belongs to the extension field GF(p^2). In this case, the Montgomery ladder treats x as though it were (x,y) where x is GF(p) but y is not. Such points belong to a "twist" group, and this group has order:

2^2 * 5 * 1526119141 * 788069478421 * 182758084524062861993 * 345246493045167733003600525204032854694l

An adversary can exploit this, by finding such invalid x that correspond to a lower order group element, and thereby try to learn partial information about a static private key used by a non-validating Montgomery ladder implementation.

A.4.4.7. Cheon security

Niche applications in ECC involve revealing points [d^e]G for one secret number d, and many different integer e, or at least one large e. One way such points could be reveal is in protocols that employ a static Diffie--Hellman oracle, a function to compute [d]P from any point P, which might be applied e times, if e is reasonably small.

Typical ECDH, to be clear, would never reveal such points, for at least two reasons:

- ECDH is ephemeral, so that the same d is never re-used across ECDH sessions (because d is used to compute [d]G and [d]Q, and then discarded),

- ECDH is hashed, so though P=[d]G is sent, the point [d]Q is hashed to get k = H([d]Q), and then [d]Q is discarded, so the fact that hash is one-way means that k should not reveal [d]Q, if k is ever somehow revealed.

The Brown--Gallant--Cheon q-1 algorithm finds d, given [d^e]G, if e|(q-1). It uses approximately sqrt(q/e) elliptic curve operations. The Cheon q+1 algorithm finds d, given all the points [d]G, [d^2]G, ..., [d^e]G, if e|(q+1), and takes a similar amount of computation. These two algorithms rely on factors e of q-1 or q+1, so the factorization of these numbers affects the security against the algorithm.

Cheon security refers to the ability to resist these algorithms.

It is possible seek out special curves with relatively high Cheon security, because q-1 and q+1 have no suitable factors e.

The curve 2y^2=x^3+x/GF(8^91+5) has typical Cheon security in terms of the factorization of q-1 and q+1. Therefore, in the niche applications that reveal the requisite points, mitigations ought to be applied, such as limiting the rate of revealing points, or using different value d as much as possible (one d per recipient).

For 2y^2=x^3+x/GF(8^91+5) the factorization of q-1 and q+1 are:

```
  q-1 = 2^3 * 101203 * 23810182454264420359 *
        109347843574637764733424980622999659259561150869769926 57
```

and

```
  q+1 = 2 * 3 * 11 * 21577 * 54829 * 392473 * 854041 *
        805420153081115125375393663558120685638177971145156481 3041
```

The q-1 and q+1 algorithms convert an oracle for function P -> [d]P into a way to find d. This may be viewed as a reduction of the discrete logarithm problem to the problem of computing the function P -> [d]P for the target d. In other words, computing P -> [d]P is almost as difficulty as solving the discrete logarithm problem. In many systems with a static Diffie--Hellman secret d, computing the function P -> [d]P needs to be difficult, or the security will be defeated. In these case, an efficient q-1 or q+1 algorithm provides a security assurance, that the computing P -> [d]P without knowing d is about as hard as solving the discrete logarithm problem.

To be completed.

A.4.4.8 Reductionist security assurance for Diffie--Hellman

A series of research work, from den Boer, from Maurer and Wolf, and from Boneh and Lipton, shows that Diffie--Hellman oracle can be used to solve a discrete logarithm, under certain conditions. In other words, the discrete logarithm problem can sometimes be reduced to the Diffie--Hellman problem.

This can be interpreted as a security assurance that Diffie--Hellman problem is at least as hard the discrete logarithm problem, albeit perhaps with some gap in the difficulty. This formalized security assurance supplements the standard conjecture that the Diffie--Hellman problem is at least as hard as the discrete logarithm. (A contrarian view is that special conditions under which such a reduction algorithm is possible might coincide with special conditions under which the discrete logarithm problem is easier.)

The general idea is to consider a Diffie--Hellman oracle in a group of order q to provide multiplication in a special representation field of order q. Recovering the ordinary field representation from the special field representation amounts to solving the discrete logarithm problem.

To recover the ordinary representation, the idea is to construct an auxiliary group of smooth order, where the group is an algebraic groups over the field of size q. Solving a discrete logarithm in the auxiliary group is possible using the Pohlig--Hellman problem, and solving the discrete logarithm in the auxiliary reveals the ordinary representation of the field, which, as already noted reveals the discrete logarithm in the original group.

The most obvious auxiliary groups have orders q-1 and q+1, but these are not smooth numbers. The next most obvious auxiliary are elliptic curve groups with complex multiplication by i, but none of these four group have smooth orders either.

A peculiar strategy to show the existence of an auxiliary group of smooth order without having any effective means of constructing the group. This can be done by finding a smooth number in the Hasse interval of q.

To be completed.

Appendix B. Test vectors

The following are some test vectors, with values explained further below.

```
0000000000000000029352b31395e382846472f782b335e783d325e79322054534554
0000000000000000000000000000000000000000000000000000000000000117
c8c0f2f404a9fabc91c939d8ea1b9e258d82e21a427b549f05c832cf8d48296ffad7
5f336f56f86de3d52b0eab85e527f2ac7b9d77605c0d5018f5faa4243fd462b1badd
fc023b3f03b469dca32446db80d9b388d753cc77aa4c3ee7e2bb86e99e7bed38f509
8c2b0d58eb27185715a48d6071657273dfbb861e515ac8bac9bfe58f2baa85908221
8c2b0d58eb27185715a48d6071657273dfbb861e515ac8bac9bfe58f2baa85908221
```

The test vectors are explained as follows. (Pseudocode generating them is supplied in Appendix C.2.)

Each line is 34 bytes, representing a non-negative 272-bit integer. The integer encoding is hexadecimal, with most significant hex digits on the left, which is to say, big-endian.

> Note: Public keys are encoded as 34-byte strings are little-endian. Encoded public keys reverse the order of the bytes found in the test vectors. The pseudocode in Appendix C.2 should make this clear: since bytes are printed in reverse order.

Each integer is either a scalar (a multiplier of curve points), or the byte representation of a point P through its x-coordinate or the x-coordinate of iP (which is the the mod 8^91+5 negation of the x-coordinate of P).

The first line is a scalar integer x. Its nonzero bytes are the ASCII representation of the string "TEST 2y^2=x^3+x/GF(8^91+5)", with the byte order reversed. As a private key, this value of x would be totally insecure, because it is too small, and like any test vector, it is public.

The second line is a representation of G, a base point on the curve.

The third line is the representation of z = xG.

The fourth and fifth lines represent updated values of x and z, obtained after application of the following 100000 scalar multiplications.

A loop of 50000 iterations is performed. Each iteration consists of two re-assignments: z = xz and x = zG via scalar multiplications. In the second assignment, the byte representation of the input point z is used as the byte representation of an scalar. Similarly, the output x is the byte representation of the point, which is will used as as the byte representation of the scalar.

The purpose of the large number of iterations is to catch a bug that has probability larger than 1/100000 of arising on pseudorandom inputs. The iterations do nothing to find rarer bugs (such as those that an adversary can invoke), or silent bugs (side channel leaks).

The sixth and seventh lines are equal to each other. As explained below, the equality of these lines represents the fact the Alice and Bob can compute the same shared DH secret. The purpose of these lines is not to catch any more bugs, but rather a sanity check that Diffie--Hellman is likely to work.

Alice initializes her DH private key to x, as already computed on the fourth line of the test vectors (which was the result of 100000 iterations). She then replaces this x by x^900 mod q (where q is the prime which is the order of the order of the base point G).

Bob sets his private key y as follows. He begins with y being the 34-byte ASCII string whose initial characters are "yet another test" (not including the quotes, of course). He then reverses the order of bytes, considers this to be a scalar, and reassigns y to yG. (So, the y on the left is new, the y on the right is old, they are not the same, after the assignment.) Another reassignment is done, as y -> yy, where the on the right side of the equation one y is treated as a scalar, the other as a point. Finally, Bob's replaces y by y^900 mod order(G), similarly to Alice's transformation.

The test code in C.2 does not compute x^900 directly. Instead it uses 900 scalar multiplication by x, to achieve multiplication by x^900. The same is done for y^900.

Both lines are xyG. The first can be computed as y(xG), and the second as x(yG). The equality of the two lines can be used to self-test an implementation, even if the implementation being tested disagrees with the test vectors above.

Appendix C. Sample code (pseudocode)

This section has sample C code that illustrates well-known elliptic algorithms, with adaptations specific to 2y^2=x^3+x/GF(8^91+5).

Warning: the sample code has not been fully hardened against side channels or any other implementation attacks; also, no independent party has reviewed the sample code.

Note: The quality of the sample code is similar to pseudocode, not reference code, or software. It compiles and runs on my personal devices, but has not otherwise been tested for quality.

Note: Non-standard C language extensions are used the sample code: the type __int128, available as an C language extension in the GNU C compiler (gcc).

Note: Non-portable C is used (beyond the non-standard C), for convenience. Two's complement integer representation of integers is assumed. Bit-shifts negative integers are used, in a way that considered non-portable under strict C, even though commonly used elsewhere.

Note: Manually minified C is used: to reduce line and character counts, and also to (arguably) aid objective code inspection by cramming as much code into a single screen and by not misleading reviewers with long comments or variable names.

Note: Automated tools, such as indent (used as in "gcc -E pseudo.c | indent"), can partially revert the C sample code spacing to a more conventional style, though other aspects of minification are not so easy to remove.

Note: The minification is not total. It tries to organize the code into meaningful units, such as placing single short functions on one line or placing all variable declarations on the same line with the function parameters. Python-like indentation is kept. (Per Lisp styling, the code clumps closing delimiters (that mainly serve the compilers.))

Note: Long sequence expressions, using the C comma operator, in place of multiple expression statements, which would be more conventional and terminated by semicolons, save some braces in control statements, such as "for" loops and "if" conditionals, and enable extra initializations in declarations.

C.1. Scalar multiplication of 34-byte strings

The sample code for scalar multiplication provides an interface for scalar multiplication. A function "mulch" takes as input 3 pointer to unsigned character strings. The first input is the location of the result, the second is the multiplier, and the third is the base point.

Note: The input ordering follows the convention of C assignment expressions z=x*y.

Note: The function name "mulch" is short for multiply character strings.

Mulch returns a Boolean value, indicating success or failure. Failure is returned only if validation is requested, and the base point is invalid.

Requesting validation is done implicitly, by comparison of pointers. Validation is requested unless the base point is the known valid base point G, or if the scalar multiple (2nd input) and the output (1st input) pointers are equal, meaning that the scalar multiple will be overwritten.

Note: The motivation here for implicitly requesting validation is that if the scalar multiple is really ephemeral, the caller should be willing, and eager, to overwrite it as soon as possible, in order to achieve forward secrecy. In this case, the need for input validation is usually negligible.

The sample code is to be considered as a single file, pseudo.c.

The file pseudo.c has two sections. The first section implements arithmetic for the field GF(8^91+5). The second section implements Montgomery's ladder for curve 2y^2=x^3+x. The two sections are not entirely independent. In particular, the field arithmetic section is not general-purpose, and could produce errors if used for different elliptic curve algorithms, such as Edwards coordinates.

Note: The scalar multiplication sample code pseudo.c file is included into 3 other sample (using a the C preprocessor directive #include "pseudo.c").

Note: Compiler optimizations make a large difference when used on the field arithmetic (for versions of the sample code where the field and curve arithmetic are in separate source files). This suggests that field arithmetic efficiency has room for further improvement by hand assembly. (The curve arithmetic might be improved by re-writing the source code.) In case, the sample code should not be considered to fully optimized.

Note: Montgomery's ladder might not be the fastest scalar multiplication algorithm for 2y^2=x^3+x/GF(8^91+5). Experimental C implementations using Bernstein's 2-D ladder algorithm (see [B1]) seem about ~10% faster . The experimental code somewhat more complicated, and thus more likely to vulnerable to side channels or overflows. Even more aggressive C code seems about ~20% faster, using Edwards coordinates, Hisil--Carter--Dawson--Wong, and Gallant--Lambert--Vanstone, and pre-computed windows (see [B4]). Again, these faster methods are more complicated, and may be more vulnerable implementation attacks. The 10% and 20% gains may be lost upon more thorough hardening against implementation attacks, or upon more thorough hand-assembly optimizations.

To be completed.

C.1.1. Field arithmetic for GF(8^91+5)

The field arithmetic sample code, is the first part of the file pseudo.c. It implements the field operations used in the Montgomery ladder algorithm for elliptic curve 2y^2=x^3+x. For example, point decompression is not used in Montgomery ladders, so the square root operation is not included the sample code. (The Legendre symbol computation is included for validation, and is quite similar to the square root operation.)

```
<CODE BEGINS>
#define RZ return z
#define F4j i j=5;for(;j--;)
#define FIX(j,r,k) q=z[j]>>r, z[j]-=q<<r, z[(j+1)%5]+=q*k
#define CMP(a,b) ((a>b)-(a<b))
#define XY(j,k) x[j]*(ii)y[k]
#define R(j,k) (zz[j]>>55*k&((k<2)*M-1))
#define MUL(m,E)\
  zz[0]= m(0,0)E(1,4)E(2,3)E(3,2)E(4,1),\
  zz[1]= m(0,1)m(1,0)E(2,4)E(3,3)E(4,2),\
  zz[2]= m(0,2)m(1,1)m(2,0)E(3,4)E(4,3),\
  zz[3]= m(0,3)m(1,2)m(2,1)m(3,0)E(4,4),\
  zz[4]= m(0,4)m(1,3)m(2,2)m(3,1)m(4,0);\
  z[0]=R(0,0)-R(4,1)*20-R(3,2)*20, z[1]=R(1,0)+R(0,1)-R(4,2)*20,\
  z[2]=R(2,0)+R(1,1)+R(0,2),       z[3]=R(3,0)+R(2,1)+R(1,2),\
  z[4]=R(4,0)+R(3,1)+R(2,2);       z[1]+=z[0]>>55; z[0]&=M-1;
typedef long long i;typedef i*f,F[5];typedef __int128 ii,FF[5];
i M=((i)1)<<55;F O={0},I={1};
f fix(f z){i j=0,q;
  for(;j<5*2;j++) FIX(j%5,(j%5<4?55:53),(j%5<4?1:-5));
  z[0]+=(q=z[0]<0)*5; z[4]+=q<<53; RZ;}
i cmp(f x,f y){i z=(fix(x),fix(y),0); F4j z+=!z*CMP(x[j],y[j]); RZ;}
f add(f z,f x,f y){F4j z[j]=x[j]+y[j]; RZ;}
f sub(f z,f x,f y){F4j z[j]=x[j]-y[j]; RZ;}
f mal(f z,i s,f y){F4j z[j]=y[j]*s; RZ;}
f mul(f z,f x,f y){FF zz; MUL(+XY,-20*XY); {F4j zz[j]=0;} RZ;}
f squ(f z,f x){mul(z,x,x); RZ;}
i inv(f z){F t;i j=272; for(mul(z,z,squ(t,z));j--;) squ(t,t);
  return mul(z,t,z), (sub(t,t,t)), cmp(O,z);}
i leg(f y){F t;i j=270; for(squ(t,squ(y,y));j--;) squ(t,t);
  return j=cmp(I,mul(y,y,t)), (sub(y,y,y),sub(t,t,t)), (2-j)%3-1;}
<CODE ENDS>
```

Field elements are stored as five-element of arrays of limbs. Each limb is an integer, possibly negative, with array z representing integer

```
  z[0] + z[1]*2^55 + z[2]*2^110 + z[3]*2^165 + z[4]*2^220
```

In other words, the radix (base) is 2^55. Say that z has m-bit limbs if each |z[i]| < 2^m.

The field arithmetic function input order follows the C assignment order, as input z=x*y, so usually the first input is the location for the result of the operation. The return value is usually just a pointer to the result's location, the first input, indicated by the preprocessor macro RZ. The functions, inv, cmp, and leg, also return an integer, which is not a field element, but usually a Boolean (or for function leg, a value in {-1,0,1}.)

The utility functions are fix and cmp. They are meant to take inputs with 58-bit limbs, and produce an output with 55-bit non-negative limbs, with the highest limb, a 53-bit value. The purpose of fix is to provide a single array representation of each field element. The function cmp fixes both its inputs, and then returns a signed comparison indicator (in {-1,0,1}).

The multiplicative functions are mul, squ, inv and leg. They are meant to take inputs with 58-bit limbs, and produce either an output with 57-bit limbs, or a small integer output. They try to do this as follows:

1. Some of the input limbs are multiplied by 20, then multiplied in pairs to 128-bit limbs, and then summed in groups of five (with at least one of the pairs having both elements not multiplied by 20). The multiplications by 20 should not cause 64-bit overflow 20*2^58 < 32*2^58=2^63, while the sums of 128-bit numbers should not cause overflow, because (1+4*20)*2^58*2^58 = 81*2^116 < 2^7*2^116 = 2^123.

2. The five 128-bit limbs are partially reduced to five 57-bit limbs. Each the five smaller limbs is obtained by summing two 55-bit limbs, extracted from sections of the 128-bit limbs, and then summing one or two much smaller values summing to less than a 55-bit limb. So, the final limbs in the multiplication are a sum of at most three 55-bit sub-limbs, making each final limb at most a 57-bit limb.

The additive functions are add, sub and mal. They are meant to take inputs with 57-bit limbs, and product an output with 58-bit limbs.

The utility and multiplicative function can be used repeatedly, because they do not lengthen the limbs.

The additive functions potentially increase the limb length, because they do not perform any reduction on the output. The additive functions should not be applied repeatedly. For example, if the output of additive additive function is fed directly as the input to an additive function, then the final output might have 59-bit limbs. In this case, if 2nd output might not be evaluated corrected if given as input to one of the multiplicative functions, an error due to overflow of 64-bit arithmetic might occur.

The lack of reduction in the additive functions trades generality for efficiency. The elliptic curve arithmetic code aims to never send the output of an additive function directly into the input of another additive function.

Note: Zeroizing temporary field values is attempted by subtracting them from themselves. Some compilers might remove these zeroization steps.

Note: The defined types f and F are essentially the equivalent. The main difference is that type F is an array, so it can be used to allocate new memory (on the stack) for a field value.

C.1.2. Montgomery ladder scalar multiplication

The second part of the file "pseudo.c" implements Montgomery's well-known ladder algorithm for elliptic curve scalar point multiplication, as it applies to the curve 2y^2=x^3+x.

The sample code, as part of the same file, is a continuation of the sample code for field arithmetic. All previous definitions are assumed.

```
<CODE BEGINS>
#define X z[0]
#define Z z[1]
enum {B=34}; typedef void _;typedef volatile unsigned char *c,C[B];
typedef F*e,E[2];typedef E*v,V[2];
f feed(f x,c z){i j=((mal(x,0,x)),B);
  for(;j--;) x[j/7]+=((i)z[j])<<((8*j)%55); return fix(x);}
c bite(c z,f x){F t;i j=((fix(mal(x,cmp(mal(t,-1,x),x),x))), B),k=5;
  for(;j--;) z[j]=x[j/7]>>((8*j)%55); {(sub(t,t,t));}
  for(;--k;) z[7*k-1]+=x[k]<<(8-k); {(sub(x,x,x));} RZ;}
i lift(e z,f x,i t){F y;return mal(X,1,x),mal(Z,1,I),t||
  -1==leg(add(y,x,mul(y,x,squ(y,x))));}
i drop(f x,e z){return inv(Z)&&mul(x,X,Z)&&(sub(X,X,X)&&sub(Z,Z,Z));}
_ let(e z,e y){i j=2;for(;j--;)mal(z[j],1,y[j]);}
_ smv(v z,v y){i j=4;for(;j--;)add(((e)z)[j],((e)z)[j],((e)y)[j]);}
v mav(v z,i a){i j=4;for(;j--;)mal(((e)z)[j],a,((e)z)[j]);RZ;}
_ due(e z){F a,b,c,d;
  mul(X,squ(a,add(a,X,Z)),mal(d,2,squ(b,sub(b,X,Z))));
  mul(Z,add(c,a,b),sub(d,a,b));}
_ ade(e z,e u,f w){F a,b,c,d;f ad=a,bc=b;
  mul(ad,add(a,u[0],u[1]),sub(d,X,Z)),
  mul(bc,sub(b,u[0],u[1]),add(c,X,Z));
  squ(X,add(X,ad,bc)),mul(Z,w,squ(Z,sub(Z,ad,bc)));}
_ duv(v a,e z){ade(a[1],a[0],z[0]);due(a[0]);}
v adv(v z,i b){V t;
  let(t[0],z[1]),let(t[1],z[0]);smv(mav(z,!b),mav(t,b));mav(t,0);RZ;}
 e mule(e z,c d){V a;E o={{1}};i
b=0,c,n=(let(a[0],o),let(a[1],z),8*B);
  for(;n--;) c=1&d[n/8]>>n%8,duv(adv(a,c!=b),z),b=c;
  let(z,*adv(a,b)); (due(*mav(a,0))); RZ;}
C G={23,1};
i mulch(c db,c d,c b){F x;E p; return
  lift(p,feed(x,b),(db==d||b==G))&&drop(x,mule(p,d))&&bite(db,x);}
<CODE ENDS>
```

This part of the sample code represents points and scalar multipliers as character strings of 34 bytes.

Note: Types c and C are used for these 34-byte encodings. Following the previous pattern for f and F, type C is an array, used for allocating new memory (on the stack) for these arrays.

The conversion functions feed and bite convert between a 34-byte string and a field value (recall, stored as five element array, base 2^55).

The conversion functions lift and drop convert between field elements and the projective line point, so that x <-> (X:1). The function lift can also test if x is the x-coordinate of the a point (x,y) on the curve 2y^2=x^3+x.

Note: Projective line points are stored in defined types e and E (for extended field element).

Note: The Montgomery ladder can implemented by working with a pair of extended field elements.

The raw scalar multiplication function "mule" takes a projective point (with defined type e), multiplies it by a scalar (encoded as byte string with defined type c), and then replaces the projective point by the multiple.

The main loop of mule is written a double-and-always-add, acting on pair projective line points. Basically it acts on the x-coordinates of the points nB and (n+1)B, for n changing.

Because the Montgomery ladder algorithm is being used, the "adv" called by mule function does nothing but swap the two values. With an appropriate isogeny, this can be viewed as addition operation.

The function "duv" called by mule, does the hard work of finding (2n)B and (2n+1)B from nB and (n+1)B. It does so, using doubling in the function "due" and differential addition, in the function "ade".

The functions "due" and "ade" are non-trivial, and use field arithmetic. They are fairly specific to 2y^2=x^3+x. They try to avoid repeated application of additive field operations.

The function smv, mav and let are more utilitarian. They are used for initialization, swapping, and zeroization.

C.1.3. Bernstein's 2-dimensional Montgomery ladder

Bernstein's 2-dimensional ladder is a variant of Montgomery's ladder that computes aP+bQ, for any two points P and Q, more quickly than computing aP and bQ separately.

Curve 2y^2=x^3+x has an efficient endomorphism, which allows a point Q = [i+1]P to compute efficiently. Gallant, Lambert and Vanstone introduced a method (now called the GLV method), to compute dP more efficiently, given such an efficient endomorphism. They write d = a + eb where e is the integer multiplier corresponding to the efficient endomorphism, and a and b are integers smaller than d. (For example, 17 bytes each instead of 34 bytes.)

The GLV method can be combined with Bernstein's 2D ladder algorithm to be applied to compute dP = (a+be)P = aP + beP = aP + bQ, where e=i+1.

This algorithm is not implemented by any pseudocode in the version the draft. (Previous versions had it.)

See [B1] for further explanation and example pseudocode.

I have estimate a ~10% speedup of this method compared to the plain Montgomery ladder. However, the code is more complicated, and potentially more vulnerable to implementation-based attacks.

C.1.4. GLV in Edwards coordinates (Hisil--Carter--Dawson--Wong)

To be completed.

It is also possible to convert to Edwards coordinates, and then use the Hisil--Wong--Carter--Dawson (HWCD) elliptic curve arithmetic.

The HWCD arithmetic can be combined with the GLV techniques to obtain a scalar multiplication potentially more efficient than Bernstein's 2-dimensional Montgomery. The downside is that it may require key-dependent array look-ups, which can be a security risk.

My implementation of this (see [B4]) gives a ~20% speed-up over my implementation of the Montgomery ladder. Of course, this speed-up may disappear upon further optimization (e.g. assembly), or further security hardening (safe table lookup code).

C.2. Sample code for test vectors

The following sample code describes the contents of a file "tv.c", with the purpose of generating the test vectors in Appendix B.

```
<CODE BEGINS>
//gcc tv.c -o tv -O3 -flto -finline-limit=200;strip tv;time ./tv
#include <stdio.h>
#include "pseudo.c"
#define M mulch
void hx(c x){i j=B;for(;j--;)printf("%02x",x[j]);printf("\n");}
int main (void){i n=1e5,j=n/2,wait=/*your mileage may vary*/7000;
  C x="TEST 2y^2=x^3+x/GF(8^91+5)",y="yet another test",z;
  M(z,x,G); hx(x),hx(G),hx(z);
  fprintf(stderr,"%30s(wait=~%ds, ymmv)","",j/wait);
  for(;j--;)if(fprintf(stderr,"\r%7d\r",j),!(M(z,x,z)&&M(x,z,G)))
   j=0*printf("Mulch fail rate ~%f :(\n",(2*j)/n);//else//debug
  fprintf(stderr,"\r%30s                \r",""),hx(x),hx(z);
  M(y,y,G);M(y,y,y);
  for(M(z,G,G),j=900;j--;)M(z,x,z);for(j=900;j--;)M(z,y,z);hx(z);
  for(M(z,G,G),j=900;j--;)M(z,y,z);for(j=900;j--;)M(z,x,z);hx(z);}
<CODE ENDS>
```

It includes the previously defined file pseudo.c, and the standard header file stdio.h.

The first for-loop in main aims to terminate in the event of the bug such that the output of mulch is an invalid value, not on the curve 2y^2=x^3+x.

Of the 100,000 scalar multiplication in this for-loop, the aim is that 50,000 include public-key validation. All 100,000 include a field-inversion, to encode points uniquely as 34-byte strings.

The second and three for-loops aims to test the compatibility with Diffie--Hellman, by showing the 900 applications of scalar multipliers x and y are the same, whether x or y is applied first.

The 1st line comment suggest possible compilation commands, with some optimization options. The run-time depends on the system, and should be slower on older and weaker systems.

Anecdotally, on a ~3 year-old personal computer, it runs in time as low as 5.7 seconds, but these were under totally uncontrolled conditions (with no objective benchmarking). (Experience has shown that on a ~10 year-old personal computer, it could be ~5 times slower.)

C.3. Sample code for a command-line demo of Diffie--Hellman

The next sample code is intended to demonstrate ephemeral (elliptic curve) Diffie--Hellman: (EC)DHE in TLS terminology.

The code can be considered as a file "dhe.c". It has both C and bash code, intermixed within comments and strings. It is bilingual: a valid bash script and valid C source code. The file "dhe.c" can be made executable (using chmod, for example), so it can be run as a bash script.

```
<CODE BEGINS>
#include "pseudo.c" /* dhe.c (also a bash script)
: demos ephemeral DH, also creates, clobbers files dhba dha dhb
: -- Dan Brown, BlackBerry, '20 */
#include <stdio.h>
_ get(c p,_*f){f&&fread ((_*)p,B,1,f)||mulch(p,p,G);}
_ put(c p,_*f){f&&fwrite((_*)p,B,1,f)&&fflush(f); bite(p,O);}
int main (_){C p="not validated",s="/dev/urandom" "\0"__TIME__;
  get(s,fopen((_*)s,"r")), mulch(p,s,G), put(p,stdout);
  get(p,stdin),            mulch(s,s,p), put(s,stderr);} /*'
[ dhe.c -nt dhe ]&&gcc -O2 dhe.c -o dhe&&strip dhe&&echo "$(<dhe.c)"
mkfifo dh{a,b,ba} 2>/dev/null || ([ ! -p dhba ] && :> dhba)
./dhe <dhba 2>dha | ./dhe >dhba 2>dhb &
sha256sum dha & sha256sum dhb  # these should be equal
(for f in dh{a,b,ba} ; do [ -f $f ] && \rm -f $f; done)# '*/
<CODE ENDS>
```

Run as a bash script, file "dhe.c" will check if it needs compile its own C code, into an executable named "dhe". Then the bash script file "dhe.c" runs the compiled executable "dhe" twice. One run is Alice's, and the other Bob's.

Each run of "dhe" generates an ephemeral secret key, by reading the file "/dev/urandom". Each run then writes to "stdout", the ephemeral public key. Each run then reads the peer's ephemeral public key from "stdin". Each run then writes to "stderr" the shared Diffie--Hellman secret. (Public-key validation is mostly unnecessary, because the ephemeral is only used once, so it is skipped by using the same pointer location for the ephemeral secret and final shared secret.)

The script "dhe.c" connects the input and output of these two using pipes. One pipe is generated by the shell command line using the shell operator "|". The other pipe is a pipe name "dhab", created with "mkfifo". The script captures the shared secrets from each run by redirecting "stderr" (as file descriptor 2), to files "dha" and "dhb", which will be made named pipes if possible.

The scripts feeds each shared secret keys into SHA-256. This demonstrates their equality. It also illustrates a typical way to use Diffie--Hellman, by deriving symmetric keys using a hash function. In multi-curve ECC, hashing a concatenation of such shared secrets (one for each curve used), could be done instead.

C.4. Sample code for public-key validation and curve basics

The next sample code demonstrates the public-key validation issues specific to 2y^2=x^3+x/GF(8^91+5). It also demonstrates the order of the curve. It also demonstrates complex multiplication by i, and the fact the 34-byte representation of points is unaffected by multiplication by i.

The code can be considered to describe a file "pkv.c". It uses the "mulch" function by including "pseudo.c".

```
<CODE BEGINS>
#include <stdio.h>
#include "pseudo.c"
#define M mulch // works with +/- x, so P ~ -P ~ iP ~ -iP
void hx(c x){i j=B;for(;j--;)printf("%02x",x[j]);printf("\n");}
int main (void){i j;// sanity check, PKV, twist insecurity demo
  C y="TEST 2y^2=x^3+x/GF(8^91+5)",z="zzzzzzzzzzzzzzzzzzzz",
  q = "\xa9\x38\x04\xb8\xa7\xb8\x32\xb9\x69\x85\x41\xe9\x2a"
  "\xd1\xce\x4a\x7a\x1c\xc7\x71\x1c\xc7\x71\x1c\xc7\x71\x1c"
  "\xc7\x71\x1c\xc7\x71\x1c\x07", // q=order(G)
  i = "\x36\x5a\xa5\x56\xd6\x4f\xb9\xc4\xd7\x48\x74\x76\xa0"
  "\xc4\xcb\x4e\xa5\x18\xaf\xf6\x8f\x74\x48\x4e\xce\x1e\x64"
  "\x63\xfc\x0a\x26\x0c\x1b\x04", // i^2=-1 mod q
  w5= "\xb4\x69\xf6\x72\x2a\xd0\x58\xc8\x40\xe5\xb6\x7a\xfc"
  "\x3b\xc4\xca\xeb\x65\x66\x66\x66\x66\x66\x66\x66\x66\x66"
  "\x66\x66\x66\x66\x66\x66\x66"; // w5=(2p+2-72q)/5
  for(j=0;j<=3;j++)M(z,(C){j},G),hx(z); // {0,1,2,3}G, but reject 0G
  M(z,q,G),hx(z); // reject qG; but qG=O, under hood:
  {F x;E p;lift(p,feed(x,G),1);mule(p,q);hx(bite(z,p[1]));}
  for(j=0;j<0*25;j++){F x;E p;lift(p,feed(x,(C){j,1}),1);mule(p,q);
  printf("%3d ",j),hx(bite(z,p[1]));}// see j=23 for choice of G
  for(j=3;j--;)q[0]-=1,M(z,q,G),hx(z);// (q-{1,2,3})G ~ {1,2,3}G
  M(z,i,G),hx(z); i[0]+=1,M(z,i,G),M(z,i,z),hx(z);// iG~G,(i+1)^2G~2G
  M(w5,w5,(C){5}),hx(w5);// twist, ord(w5)=5, M(z,z,p) skipped PKV(p)
  M(G,(C){1},w5),hx(G);// reject w5 (G unch.); but w5 leaks z mod 5:
  for(j=10;j--;)M(z,y,G),z[0]+=j,M(z,z,w5),hx(z);}
<CODE ENDS>
```

The sample code demonstrates the need for public-key validation even when using the Montgomery ladder for scalar multiplication. It does this by finding points of low order on the twist of the curve. This invalid points can leak bits of the secret multiplier. This is because the curve 2y^2=x^3+x/GF(8^91+5) is not fully "twist secure". (Its twist security is typical of that of a random curve.)

Appendix D. Minimizing trapdoors and backdoors

The main advantage of curve 2y^2=x^3+x/GF(8^91+5) over almost all other elliptic curves is its Kolmogorov complexity is almost minimal among curves of sufficient resistance to the Pollard rho attack on the discrete logarithm problem.

See [AB] and [B1] for some details.

D.1. Decimal exponential complexity

The curve can be described with 21 characters:

```
2  y  ^  2  =  x  ^  3  +  x  /  G  F  (  8  ^  9  1  +  5  )
1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21
```

Those familiar with ECC will recognize that these 21 characters suffice to specify the curve up to the level of detail needed to describe the cost of the Pollard rho algorithm, as well as many other security properties (especially resistance to other known attacks on the discrete logarithm problem, such as Pohlig--Hellman and Menezes--Okamoto--Vanstone).

> Note: The letters GF mean Galois Field, and are quite traditional mathematics, and every elliptic curve in cryptographic needs to use some notation for the finite field.

We may therefore describe the curve's Kolmogorov complexity as 21 characters.

> Note: The idea of low Kolmogorov complexity is hard to specify exactly. Nonetheless, a claim of nearly minimal Kolmogorov complexity is quite falsifiable. The falsifier need merely specify several other (secure) elliptic curves using 21 or fewer characters. (But if the other curves use a different specification language, then a fair comparison should re-specify 2y^2=x^3+x/GF(8^91+5) in this specification language.)

D.1.1. A shorter isomorphic curve

The curve is isomorphic to a curve specifiable in 20 characters:

y^2=x^3-x/GF(8^91+5)

Generally, isomorphic curves have essentially equivalently hard discrete logarithm problems, so one could argue that curve 2y^2=x^3+x/GF(8^91+5) could be rated as having Kolmogorov complexity at most 20 characters.

Isomorphic curves, however, may differ slightly in security, due to issues of efficiency, and implementability. The 21-character specification uses an equation in Montgomery form, which creates an incentive to use the Montgomery ladder algorithm, which is both safe and efficient [Bernstein?].

D.1.2. Other short curves

Allowing for non-prime fields, then the binary-field curve known as sect283k1 has a 22-character description:

y^2+xy=x^3+1/GF(2^283)

This curve was formerly one of the fifteen curves recommended by NIST. Today, a binary curve is curve is considered risky, due to advances in elliptic curve discrete logarithm problem over extension fields, such as recent asymptotic advances on discrete logarithms in low-characteristic fields [HPST] and [Nagao]. According to [Teske], some characteristic-two elliptic curves could be equipped with a secretly embedded backdoor (but sect283k1's short description should help mitigate that risk).

This has a longer overall specification than curve 2y^2=x^3+x/GF(8^91+5), but the field part is shorter field specification. Perhaps an isomorphic curve can be found (one with three terms), so that total length is 21 or fewer characters.

A non-prime field tends to be slower in software. A non-prime field is therefore perhaps riskier due to some recent research on attacking non-prime field discrete logarithms and elliptic curves,

D.1.3. Converting DEC characters to bits

The units of characters as measuring Kolmogorov complexity is not calibrated as bits of information. Doing so formally would be very difficult, but the following approach might be reasonable.

Set the criteria for the elliptic curve. For example, e.g. prime field, size, resistance (of say 2^128 bit operations) to known attacks on the discrete logarithm problem (Pollard rho, MOV, etc.). Then list all the possible ECC curve specification with Kolmogorov complexity of 21 characters or less. Take the base two logarithm of this number. This is then an calibrated estimate of the number of bits needed to specify the curve. It should be viewed as a lower bound, in case some curves were missed.

To be completed.

D.1.4. Common acceptance of decimal exponential notation

The decimal exponentiation notation used in to measure decimal exponential complexity is quite commonly accepted, almost standard, in mathematical computer programming.

For example, as evidence of this common acceptance, here is a slightly edited session of the program "bc" (versions of which are standardized in POSIX).

```
<CODE BEGINS>
$ BC_LINE_LENGTH=71 bc
bc 1.06.95
Copyright ... Free Software Foundation, Inc.
...
  p=8^91+5 ; p; obase=16; p

151771007205135083665582961470587414581438034300948400097797844510851\
 89728165691397
2000000000000000000000000000000000000000000000000000000000000000000005
  define v(b,e,m){
    auto a; for(a=1;e>0;e/=2){
    if(e%2==1) {a=(a*b)%m;}
    b=(b^2)%m;}
  return(a);}
  v(571,p-1,p)
1
  x = (1*256) + (23*1)
  v(2*(x^3+x),(p-1)/2,p)
1
  y = (((p+1)/2)*v(2*(x^3+x),(p+3)/8,p))%p
  (2*y^2)%p == (x^3+x)%p
1
  (2*y^2 -(x^3+x))%(8^91+5)
0
<CODE ENDS>
```

Note: Input lines have been indented at least two extra spaces, and can be pasted into a "bc" session. (Pasting the output lines causes a few spurious results.)

The sample code demonstrates that "bc" directly accepts the notations "8^91+5" and "x^3+x": parts parts of the curve specification "2y^2=x^3+x/GF(8^91+5)", which goes to show how much of the notation used in this specification is commonly accepted.

Note: Defined function "v" implements modular exponentiation, with returning v(b,e,m) returning (b^e mod m). Then, "v" is used to show that p=8^91+5 is a Fermat pseudoprime to base 571 (evidence that p is prime). The value x defined is the x-coordinate of the recommend base point G. Then, another computation with "v" shows that 2(x^3+x) has Legendre symbol 1, which implies (assuming p is prime) that there exists y with 2y^2=x^3+x, namely y = (1/2)sqrt(2(x^3+x)). The value of y is computed, again using "v" (but also a little luck). The curve equation is then tested twice with two different expressions, somewhat similar to the mathematical curve specification 2y^2=x^3+x/GF(8^91+5).

D.2. General benefits of low Kolmogorov complexity to ECC

The intuitive benefit of low Kolmogorov complexity to cryptography is well known, but very informal and heuristic. The general benefit is believed to be a form of subversion-resistance, where the attacker is the designer of the cryptography. The idea is that low Kolmogorov complexity thwarts that type of subversion which causes high Kolmogorov complexity. Exhaustive searches for weaknesses would seem to require relatively high Kolmogorov complexity, compared to lowest complexity non-weak examples in the search.

Often, fixed numbers in cryptographic algorithms with low Kolmogorov complexity are called "nothing-up-my-sleeve" numbers. (Bernstein et al. uses terms in "rigid", for a very similar idea, but with an emphasis on efficiency instead of compressibility.)

For elliptic curves, the informal benefit may be stated as the following gains.

- Low Kolmogorov complexity defends against insertion of a keyed trapdoor, meaning the curve can broken using a secret trapdoor, by an algorithm (eventually discovered by the public at large). For example, the Dual EC DRBG is known to capable of having such a trapdoor. Such a trapdoor would information-theoretically imply an amount of information, comparable the size of the secret, to be embedded in the curve specification. If the calibrated estimate for the number of bits is sufficiently accurate, then such a key cannot be large.

- Low Kolmogorov complexity defends against a secret attack (presumably difficult to discover), which affects a subset of curves such that (a) whether or not a specific curve is affected is a somewhat pseudorandom function of its natural specification, and (b) the probably of a curve being affected (when drawn uniformly from some sensible of curve specification), is low. For an example of real-world attacks meeting the conditions (a) and (b) consider the MOV attack. Exhaustively finding curve meeting these two conditions is likely to result in relatively high Kolmogorov complexity. The supply of low Kolmogorov complexity curves is so low that the probability of any falling into the weak class is low.

- Even more hypothetically, there may yet exist undisclosed classes of weak curves, or attacks, which 2y^2=x^3+x/GF(8^91+5) avoids somehow. A real-world example is prime-order, or low cofactor curves, which are rare among all curves, but which better resist the Pohlig--Hellman attack. If this happens, then it should be considered a fluke.

Low Kolmogorov complexity is not a panacea. The worst failure would be attacks that increase in strength as Kolmogorov complexity gets lower. Two examples illustrate this strongly.

D.2.1. Precedents of low Kolmogorov complexity in ECC

To be completed.

Basically, the curves sect283k1, Curve25519, and Brainpool curves can be argued as mitigating the risk of manipulated designed-in weakness, by virtue of the low Kolmogorov complexity.

To be completed.

D.3. Risks of low Kolmogorov complexity

Low Kolmogorov complexity is not a panacea for cryptography.

Indeed, it may even add its own risks, if some weakness are positively correlated with low Kolmogorov complexity, making some attacks stronger.

In other words, choosing low Kolmogorov complexity might just accidentally weaken the cryptography. Or worse, if attackers find and hold secret such weaknesses, then attackers can intentionally include the weakness, by using low Kolmogorov serving as a cover, thereby subverting the algorithm.

Evidence of positive correlations between curve weakness and low Kolmogorov complexity might help assess this risk.

In general cryptography (not ECC), the shortest cryptography algorithms may be the least secure, such as the identity function as an encryption function.

Within ECC, however, some minimum threshold of complexity must be met for interoperability. But curve size is positively correlated with security (via Pollard rho) and negatively correlated with complexity (at least for fields, larger fields needs larger specifications). Therefore, there is a somewhat negative correlation between Pollard rho security of ECC and Kolmogorov complexity of the field size.

Beyond field size in ECC, there is some negative correlations in the curve equation.

Singular cubics have equations that look very similar to those commonly used elliptic curves. For smooth singular curves (irreducible cubics) a group can be defined, using more or less the same arithmetic as for a elliptic curve. For example y^2=x^3/GF(8^91+5) is such a cubic. The resulting group has an easy discrete logarithm problem, because it can be mapped to the field.

Supersingular elliptic curves can also be specified with low Kolmogorov complexity, and these are vulnerable to MOV attack, another negative correlation.

Combining the above, a low Kolmogorov complexity elliptic curve, y^2=x^3+1/GF(2^127-1), with 21-character decimal exponential complexity, suffers from three well-known attacks:

1. The MOV (Menezes--Okamato--Vanstone) attack.

2. The Pohlig--Hellman attack (since it has 2^127 points).

3. The Pollard rho attack (taking 2^63 steps, instead of the 2^126 of exhaustive).

Had all three attacks been unknown, an implementer seeking low Kolmogorov complexity, might have been drawn to curve y^2=x^3+1/GF(2^127-1). (This document's curve 2y^2=x^3+x/GF(8^91+5) uses 1 more character and is much slower since, the field size has twice as many bits.)

Had an attacker known one of three attacks, the attacker could found y^2=x^3+1/GF(2^127-1), proposed it, touted its low Kolmogorov complexity, and maybe successfully subverted the system security.

Note: The curve y^2=x^3+1/GF(2^127-1) not only has low decimal exponential complexity, it also has high efficiency: fast field arithmetic and fairly fast curve arithmetic (for its bit lengths). So high efficiency can also be positively correlated with weakness.

It can be argued, that pseudorandomized curves, such as NIST P-256 and Brainpool curves, are an effective way mitigate such attacks positively correlated with low complexity. More precisely, strong pseudorandomization somewhat mitigates the attacker's subversion ability, by reducing an easy look up of the weakest curve to an exhaustive search by trial and error, intuitively implying a probable high Kolmogorov complexity (proportional the rarity of the weakness).

It can be further argued that all major known weak classes of curves in ECC are positively correlated with low complexity, in that the weakest curves have very low complexity. No major known weak classes of curves imply an increase in Kolmogorov complexity, except perhaps Teske's class of curves.

In defense of low complexity, it can be argued that the strongest way to resist secret attacks is to find the attacks.

For these reasons, this specification suggests to use curve 2y^2=x^3+x/GF(8^91+5) in multi-curve elliptic curve cryptography, in combination with at least one pseudo-randomized curve.

To be completed.

D.4. Alternative measures of Kolmogorov complexity

Decimal exponential complexity arguably favors decimal and the exponentiation operators, rather than the arbitrary notion of compressibility.

Allowing more arbitrary compression schemes introduces another possible level of complexity, the compression scheme itself, somewhat defeating the purpose of nothing-up-sleeve number. An attacker might be able to choose a compression scheme among many that somehow favors a weak curve.

Despite this potential extra complexity, one can still seek a measure more objective than decimal complexity. To this end, in [B3], I adapted the Godel's approach for general recursive functions, which breaks down all computation into succession, composition, repetition, and minimization.

The adaption is a miniature programming language called Roll to describe number-related functions, including constant functions. A Roll program for the constant function that always return 8^91+5 is:

```
<CODE BEGINS>
8^91+5 subs 8^91+1 in +4
8^91+1 subs 2^273 in +1
2^273 subs 273 in 2^
273 subs 17 in *16+1
17  subs  1 in *16+1
*16+1 roll +16 up 1
+16 subs +8 in +8
+8  subs +4 in +4
+4  subs +2 in +2
2^ roll *2 up  1
1 subs  in +2
*2 roll +2 up  0
+2 subs +1 in +1
0 subs   in +1
<CODE ENDS>
```

A Roll program has complexity measured in its length in number of words (space-separated substrings). This program has 68 words. Constants (e.g. field sizes) can be compared using roll complexity, the shortest known length of their implementations in Roll.

In [B3], several other ECC field sizes are given programs. The only prime field size implemented with 68 or fewer words was 2^521-1. (The non-prime field size (2^127-1)^2 has 58-word "roll" program.) Further programming effort might produce shorter programs.

Note: Roll programs have a syntax implying some redundancy. Further work may yet establish a reasonable normalization for roll programs, resulting in a more calibrated complexity measure in bits, making the units closed to a universal kind of Kolmogorov complexity.

Appendix E. Primality proofs and certificates

Recent work of Albrecht and others [AMPS] has shown the combination of an adversarially chosen prime, and users using improper probabilistic primality tests can make user vulnerable to an attack.

The adversarial primes in their attack are typically the result of an exhaustive search. These bad primes would therefore typically contain an amount of information corresponding to the length of their search, putting a predictable lower bound on their Kolmogorov complexity.

The two primes involved for 2y^2=x^3+x/GF(8^91+5) should perhaps already resist [AMPS] because of the following compact representation of these primes:

```
p = 8^91+5
q = #(2y^2=x^3+x/GF(8^91+5))/72
```

This attack [AMPS] can also be resisted by:

- properly implementing probabilistic primality test, or
- implementing provable primality tests.

Provable primality tests can be very slow, but can be separated into two steps:

-- a slow certificate generation, and

-- a fast certificate verification.

The certificate is a set of data, representing an intermediate step in the provable primality test, after which the completion of the test is quite efficient.

Pratt primality certificate generation for any prime p, involves factorizing p-1, which can be very slow, and then recursively generating a Pratt primality certificate for each prime factor of p-1. Essentially, each prime has a unique Pratt primality certificate.

Pratt primality certificate verification of (p-1), involves search for g such that 1 = (g^(p-1) mod p) and 1 < (g^((p-1)/q) mod p) for each q dividing p-1, and then recursively verifying each Pratt primality certificate for each prime factor q of p-1.

In this document, we specify a Pratt primality certificate as a sequence of (candidate) primes each being 1 plus a product of previous primes in the list, with certificate stating this product.

Although Pratt primality certificate verification is quite efficient, an ECC implementation can opt to trust 8^91+5 by virtue of verifying the certificate once, perhaps before deployment or compile time.

E.1. Pratt certificate for the field size 8^91+5

Define 52 positive integers, a,b,c,...,z,A,...,Z as follows:

a=2 b=1+a c=1+aa d=1+ab e=1+ac f=1+aab g=1+aaaa h=1+abb i=1+ae
j=1+aaac k=1+abd l=1+aaf m=1+abf n=1+aacc o=1+abg p=1+al q=1+aaag
r=1+abcc s=1+abbbb t=1+aak u=1+abbbc v=1+ack w=1+aas x=1+aabbi
y=1+aco z=1+abu A=1+at B=1+aaaadh C=1+acu D=1+aaav E=1+aeff F=1+aA
G=1+aB H=1+aD I=1+acx J=1+aaacej K=1+abqr L=1+aabJ M=1+aaaaaabdt
N=1+abdpw O=1+aaaabmC P=1+aabeK Q=1+abcfgE R=1+abP S=1+aaaaaaabcM
T=1+aIO U=1+aaaaaduGS V=1+aaaabbnuHT W=1+abffLNQR X=1+afFW
Y=1+aaaaauX Z=1+aabzUVY.

Note: variable concatenation is used to indicate multiplication. For example, f = 1+aab = 1+2*2*(1+2) = 13.

Note: One must verify that Z=8^91+5.

Note: The Pratt primality certificate involves finding a generator g for each the prime (after the initial prime). It is possible to list these in the certificate, which can speed up verification by a small factor.

(2,b), (2,c), (3,d), (2,e), (2,f), (3,g), (2,h), (5,i), (6,j),
(3,k), (2,l), (3,m), (2,n), (5,o), (2,p), (3,q), (6,r), (2,s),
(2,t), (6,u), (7,v), (2,w), (2,x), (14,y),(3,z), (5,A), (3,B),
(7,C), (3,D), (7,E), (5,F), (2,G), (2,H), (2,I), (3,J), (2,K),
(2,L),(10,M), (5,N), (10,O),(2,P), (10,Q),(6,R), (7,S), (5,T),
(3,U), (5,V), (2,W), (2,X), (3,Y), (7,Z).

Note: The decimal values for a,b,c,...,Y are given by: a=2, b=3, c=5, d=7, e=11, f=13, g=17, h=19, i=23, j=41, k=43, l=53, m=79, n=101, o=103, p=107, q=137, r=151, s=163, t=173, u=271, v=431, w=653, x=829, y=1031, z=1627, A=2063, B=2129, C=2711, D=3449, E=3719, F=4127, G=4259, H=6899, I=8291, J=18041, K=124123, L=216493, M=232513, N=2934583, O=10280113, P=16384237, Q=24656971, R=98305423, S=446424961, T=170464833767, U=115417966565804897, V=4635260015873357770993, W=156151230751602494064296769877 9, X=167553393621084508180871720014384259, Y=145302302948204485494451955596474029 4049.

E.2. Pratt certificate for subgroup order

Define 56 variables a,b,...,z,A,B,...,Z,!,@,#,$, with new values:

a=2 b=1+a c=1+a2 d=1+ab e=1+ac f=1+a2b g=1+a4 h=1+ab2 i=1+ae j=1+a2d k=1+a3c l=1+abd m=1+a2f n=1+acd o=1+a3b2 p=1+ak q=1+a5b r=1+a2c2 s=1+am t=1+ab2d u=1+abi v=1+ap w=1+a2l x=1+abce y=1+a5e z=1+a2t A=1+a3bc2 B=1+a7c C=1+agh D=1+a2bn E=1+a7b2 F=1+abck G=1+a5bf H=1+aB I=1+aceg J=1+a3bc3 K=1+abA L=1+abD M=1+abcx N=1+acG O=1+aqs P=1+aqy Q=1+abrv R=1+ad2eK S=1+a3bCL T=1+a2bewM U=1+aijsJ V=1+auEP W=1+agIR X=1+a2bV Y=1+a2cW Z=1+ab3oHOT !=1+a3SUX @=1+abNY! #=1+a4kzF@ $=1+a3QZ#

Note: numeral after variable names represent powers. For example, f = 1 + a2b = 1 + 2^2 * 3 = 13.

The last variable, $, is the order of the base point, and the order of the curve is 72$.

Note: Punctuation used for variable names !,@,#,$, would not scale for larger primes. For larger primes, a similar format might work by using a prefix-free set of multi-letter variable names. E.g. replace, Z,!,@,#,$ by Za,Zb,Zc,Zd,Ze:

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