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Pairing-Friendly Curves

Abstract

Pairing-based cryptography, a subfield of elliptic curve cryptography, has received attention due to its flexible and practical functionality. Pairings are special maps defined using elliptic curves and it can be applied to construct several cryptographic protocols such as identity-based encryption, attribute-based encryption, and so on. At CRYPTO 2016, Kim and Barbulescu proposed an efficient number field sieve algorithm named exTNFS for the discrete logarithm problem in a finite field. Several types of pairing-friendly curves such as Barreto-Naehrig curves are affected by the attack. In particular, a Barreto-Naehrig curve with a 254-bit characteristic was adopted by a lot of cryptographic libraries as a parameter of 128-bit security, however, it ensures no more than the 100-bit security level due to the effect of the attack. In this memo, we list the security levels of certain pairing-friendly curves, and motivate our choices of curves. First, we summarize the adoption status of pairing-friendly curves in standards, libraries and applications, and classify them in the 128-bit, 192-bit, and 256-bit security levels. Then, from the viewpoints of "security" and "widely used", we select the recommended pairing-friendly curves considering exTNFS.

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1. Introduction

1.1. Pairing-based Cryptography

Elliptic curve cryptography is an important area in currently deployed cryptography. The cryptographic algorithms based on elliptic curve cryptography, such as the Elliptic Curve Digital Signature Algorithm (ECDSA), are widely used in many applications.

Pairing-based cryptography, a subfield of elliptic curve cryptography, has attracted much attention due to its flexible and practical functionality. Pairings are special maps defined using elliptic curves. Pairings are fundamental in the construction of several cryptographic algorithms and protocols such as identity-based encryption (IBE), attribute-based encryption (ABE), authenticated key exchange (AKE), short signatures, and so on. Several applications of pairing-based cryptography are currently in practical use.

As the importance of pairings grows, elliptic curves where pairings are efficiently computable are studied and the special curves called pairing-friendly curves are proposed.

1.2. Applications of Pairing-based Cryptography

Several applications using pairing-based cryptography have already been standardized and deployed. We list here some examples of applications available in the real world.

IETF published RFCs for pairing-based cryptography such as Identity-Based Cryptography [RFC5091], Sakai-Kasahara Key Encryption (SAKKE) [RFC6508], and Identity-Based Authenticated Key Exchange (IBAKE) [RFC6539]. SAKKE is applied to Multimedia Internet KEYing (MIKEY) [RFC6509] and used in 3GPP [SAKKE].

Pairing-based key agreement protocols are standardized in ISO/IEC [ISOIEC11770-3]. In [ISOIEC11770-3], a key agreement scheme by Joux

[Joux00], identity-based key agreement schemes by Smart-Chen-Cheng [CCS07] and Fujioka-Suzuki-Ustaoglu [FSU10] are specified.

MIRACL implements M-Pin, a multi-factor authentication protocol [M-Pin]. The M-Pin protocol includes a type of zero-knowledge proof, where pairings are used for its construction.

The Trusted Computing Group (TCG) specified the Elliptic Curve Direct Anonymous Attestation (ECDAA) in the specification of a Trusted Platform Module (TPM) [TPM]. ECDAA is a protocol for proving the attestation held by a TPM to a verifier without revealing the attestation held by that TPM. Pairings are used in the construction of ECDAA. FIDO Alliance [FIDO] and W3C [W3C] also published an ECDAA algorithm similar to TCG.

Intel introduced Intel Enhanced Privacy ID (EPID) that enables remote attestation of a hardware device while preserving the privacy of the device as part of the functionality of Intel Software Guard Extensions (SGX) [EPID]. They extended TPM ECDAA to realize such functionality. A pairing-based EPID was proposed [BL10] and distributed along with Intel SGX applications.

Zcash implemented their own zero-knowledge proof algorithm named Zero-Knowledge Succinct Non-Interactive Argument of Knowledge (zk-SNARKs) [Zcash]. zk-SNARKs are used for protecting the privacy of transactions of Zcash. They use pairings to construct zk-SNARKs.

Cloudflare introduced Geo Key Manager [Cloudflare] to restrict distribution of customers' private keys to a subset of their data centers. To achieve this functionality, ABE is used, and pairings take a role as a building block. In addition, Cloudflare published a new cryptographic library, the Cloudflare Interoperable, Reusable Cryptographic Library (CIRCL) [CIRCL] in 2019. They plan to include securely implemented subroutines for pairing computations on certain secure pairing-friendly curves in CIRCL.

Currently, Boneh-Lynn-Shacham (BLS) signature schemes are being standardized [I-D.boneh-bls-signature] and utilized in several blockchain projects such as Ethereum [Ethereum], Algorand [Algorand], Chia Network [Chia], and DFINITY [DFINITY]. The aggregation functionality of BLS signatures is effective for their applications of decentralization and scalability.

1.3. Motivation and Contribution

At CRYPTO 2016, Kim and Barbulescu proposed an efficient number field sieve (NFS) algorithm for the discrete logarithm problem in a finite field GF(p^k) [KB16]. The attack improves the polynomial selection that is the first step in the number field sieve algorithm for discrete logarithms in GF(p^k). The idea is applicable when the embedding degree k is a composite that satisfies k = i*j (gcd (i, j) = 1, i, j> 1). The basic idea is based on the equality GF(p^k) = (GF(p^i)^j) and one of the improvement for reducing the amount of cost for solving the discrete logarithm problem is using sub-field calculation. Several types of pairing-friendly curves such as Barreto-Naehrig curves (BN curves)[BN05] and Barreto-Lynn-Scott curves (BLS curves)[BLS02] are affected by the attack, since a pairing-friendly curve suitable for cryptographic applications requires that the discrete logarithm problem is sufficiently difficult. Please refer to [KB16] for detailed ideas and calculation algorithms of the attack by Kim. In particular, BN254, which is a BN curve with a 254-bit characteristic effective for pairing calculations, was adopted by a lot of cryptographic libraries as a parameter of the 128-bit security level, however, BN254 ensures no more than the 100-bit security level due to the effect of the attack, where the security levels described in this memo correspond to the security strength of NIST recommendation [NIST].

To resolve this effect immediately, several research groups and implementers re-evaluated the security of pairing-friendly curves and they respectively proposed various curves that are secure against the attack [BD18] [BLS12-381].

In this memo, we list the security levels of certain pairing-friendly curves, and motivate our choices of curves. First, we summarize the adoption status of pairing-friendly curves in international standards, libraries and applications, and classify them in the 128-bit, 192-bit, and 256-bit security levels. Then, from the viewpoints of "security" and "widely used", pairing-friendly curves corresponding to each security level are selected in accordance with the security evaluation by Barbulescu and Duquesne [BD18].

As a result, we recommend the BLS curve with 381-bit characteristic of embedding degree 12 and the BN curve with the 462-bit characteristic for the 128-bit security level, and the BLS curves of embedding degree 48 with the 581-bit characteristic for the 256-bit security level. This memo shows their specific test vectors.

1.4. Requirements Terminology

The key words "MUST", "MUST NOT", "REQUIRED", "SHALL", "SHALL NOT", "SHOULD", "SHOULD NOT", "RECOMMENDED", "NOT RECOMMENDED", "MAY", and "OPTIONAL" in this document are to be interpreted as described in BCP 14 [RFC2119] [RFC8174] when, and only when, they appear in all capitals, as shown here.

2. Preliminaries

2.1. Elliptic Curves

Let p be a prime number and q = p^n for a natural number n > 0, where p at least 5. Let GF(q) be a finite field. The curve defined by the following equation E is called an elliptic curve:

E : y^2 = x^3 + a * x + b,

and a and b in GF(q) satisfy the discriminant inequality 4 * a^3 + 27 * b^2 != 0 mod q. This is called the Weierstrass normal form of an elliptic curve.

A solution (x,y) to the equation E can be thought of as a point on the corresponding curve. For a natural number k, we define the set of (GF(q^k))-rational points of E, denoted by E(GF(q^k)), to be the set of all solutions (x,y) in GF(q^k), together with a 'point at infinity' O_E, which is defined to lie on every vertical line passing through the curve E.

The set E(GF(q^k)) forms a group under a group law that can be defined geometrically as follows. For P and Q in E(GF(q^k)) define P + Q to be the reflection around the x-axis of the unique third point R of intersection of the straight line passing through P and Q with the curve E. If the straight line is tangent to E, we say that it passes through that point twice. The identity of this group is the point at infinity O_E. We also define scalar multiplication [K]P for a positive integer K as the point P added to itself (K-1) times. Here, [0]P becomes the point at infinity O_E and the relation [-K]P = -([K]P) is satisfied.

2.2. Pairings

A pairing is a bilinear map defined on two subgroups of rational points of an elliptic curve. Examples include the Weil pairing, the Tate pairing, the optimal Ate pairing [Ver09], and so on. The optimal Ate pairing is considered to be the most efficient to compute and is the one that is most commonly used for practical implementation.

Let E be an elliptic curve defined over a prime field GF(p). Let k be the minimum integer for which r is a divisor of p^k - 1; this is called the embedding degree of E over GF(p). Let G_1 be a cyclic subgroup of E(GF(p)) of order r, there also exists a cyclic subgroup of E(GF(p^k)) of order r, define this to be G_2. Let d be a divisor of k and E' be an elliptic curve defined over GF(p^(k/d)). If an isomorphism from E' to E(GF(p^k)) exists, then E' is called the twist of E. It can sometimes be convenient for efficiency to do the computations of G_2 in the twist E', and so consider G_2 to instead be a subgroup of E'. Let G_T be an order r subgroup of the multiplicative group (GF(p^k))^*; this exists by definition of k.

A pairing is defined as a bilinear map e: (G_1, G_2) -> G_T satisfying the following properties:

1. Bilinearity: for any S in G_1, T in G_2, and integers K and L, e([K]S, [L]T) = e(S, T)^{K * L}.

2. Non-degeneracy: for any T in G_2, e(S, T) = 1 if and only if S = O_E. Similarly, for any S in G_1, e(S, T) = 1 if and only if T = O_E.

In applications, it is also necessary that for any S in G_1 and T in G_2, this bilinear map is efficiently computable.

We define some of the terminology used in this memo as follows:

GF(p): a finite field with characteristic p.

GF(p^k): an extension field of degree k.

(GF(p))*: a multiplicative group of GF(p).

(GF(p^k))*: a multiplicative group of GF(p^k).

b: a primitive element of the multiplicative group (GF(p))^*.

O_E: the point at infinity over an elliptic curve E.

E(GF(p^k)): the group of GF(p^k)-rational points of E.

#E(GF(p^k)): the number of GF(p^k)-rational points of E.

r: the order of G_1 and G_2.

BP: a point in G_1. (The 'base point' of a cyclic subgroup of G_1)

h: the cofactor h = #E(GF(p)) / r, where gcd(h, r)=1.

2.3. Barreto-Naehrig Curves

A BN curve [BN05] is a family of pairing-friendly curves proposed in 2005. A pairing over BN curves constructs optimal Ate pairings.

A BN curve is defined by elliptic curves E and E' parameterized by a well-chosen integer t. E is defined over GF(p), where p is a prime number and at least 5, and E(GF(p)) has a subgroup of prime order r. The characteristic p and the order r are parameterized by

```
p = 36 * t^4 + 36 * t^3 + 24 * t^2 + 6 * t + 1
r = 36 * t^4 + 36 * t^3 + 18 * t^2 + 6 * t + 1
```

for an integer t.

The elliptic curve E has an equation of the form E: y^2 = x^3 + b, where b is a primitive element of the multiplicative group (GF(p))^* of order (p - 1).

In the case of BN curves, we can use twists of the degree 6. If m is an element that is neither a square nor a cube in an extension field GF(p^2), the twist E' of E is defined over an extension field GF(p^2) by the equation E': y^2 = x^3 + b' with b' = b / m or b' = b * m. BN curves are called D-type if b' = b / m, and M-type if b' = b * m. The embedding degree k is 12.

A pairing e is defined by taking G_1 as a subgroup of E(GF(p)) of order r, G_2 as a subgroup of E'(GF(p^2)), and G_T as a subgroup of a multiplicative group (GF(p^12))^* of order r.

2.4. Barreto-Lynn-Scott Curves

A BLS curve [BLS02] is a another family of pairing-frinedly curves proposed in 2002. Similar to BN curves, a pairing over BLS curves constructs optimal Ate pairings.

A BLS curve is defined by elliptic curves E and E' parameterized by a well-chosen integer t. E is defined over a finite field GF(p) by an equation of the form E: y^2 = x^3 + b, and its twist E': y^2 = x^3 + b', is defined in the same way as BN curves. In contrast to BN curves, E(GF(p)) does not have a prime order. Instead, its order is divisible by a large parameterized prime r and denoted by h * r with cofactor h. The pairing is defined on the r-torsion points. In the same way as BN curves, BLS curves can be categorized as D-type and M-type.

BLS curves vary in accordance with different embedding degrees. In this memo, we deal with the BLS12 and BLS48 families with embedding degrees 12 and 48 with respect to r, respectively.

In BLS curves, parameters p and r are given by the following equations:

```
BLS12:
    p = (t - 1)^2 * (t^4 - t^2 + 1) / 3 + t
    r = t^4 - t^2 + 1
BLS48:
    p = (t - 1)^2 * (t^16 - t^8 + 1) / 3 + t
    r = t^16 - t^8 + 1
```

for a well chosen integer t where t must be 1 (mod 3).

A pairing e is defined by taking G_1 as a subgroup of E(GF(p)) of order r, G_2 as an order r subgroup of E'(GF(p^2)) for BLS12 and of E'(GF(p^8)) for BLS48, and G_T as an order r subgroup of a multiplicative group (GF(p^12))^* for BLS12 and of a multiplicative group (GF(p^48))^* for BLS48.

2.5. Representation Convention for an Extension Field

Pairing-friendly curves use a tower of some extension fields. In order to encode an element of an extension field, focusing on interoperability, we adopt the representation convention shown in Appendix J.4 of [I-D.ietf-lwig-curve-representations] as a standard and effective method. Note that the big-endian encoding is used for an element in GF(p) which follows to mcl [mcl], ISO/IEC 15946-5 [ISOIEC15946-5] and etc.

Let GF(p) be a finite field of characteristic p and GF(p^d) = GF(p)(i) be an extension field of GF(p) of degree d.

For an element s in GF(p^d) such that s = s_0 + s_1 * i + ... + s_{d - 1} * i^{d - 1} where s_0, s_1, ... , s_{d - 1} in the basefield GF(p), s is represented as octet string by oct(s) = s_0 || s_1 || ... || s_{d - 1}.

Let GF(p^d') = GF(p^d)(j) be an extension field of GF(p^d) of degree d' / d.

For an element s' in GF(p^d') such that s' = s'_0 + s'_1 * j + ... + s'_{d' / d - 1} * j^{d' / d - 1} where s'_0, s'_1, ..., s'_{d' / d - 1} in the basefield GF(p^d), s' is represented as integer by oct(s') = oct(s'_0) || oct(s'_1) || ... || oct(s'_{d' / d - 1}), where oct(s'_0), ... , oct(s'_{d' / d - 1}) are octet strings encoded by above convention.

In general, one can define encoding between integer and an element of any finite field tower by inductively applying the above convention.

The parameters and test vectors of extension fields described in this memo are encoded by this convention and represented in an octet stream.

When applications communicate elements in an extension field, using the compression method [MP04] may be more effective. In that case, care for interoperability must be taken.

3. Security of Pairing-Friendly Curves

3.1. Evaluating the Security of Pairing-Friendly Curves

The security of pairing-friendly curves is evaluated by the hardness of the following discrete logarithm problems:

* The elliptic curve discrete logarithm problem (ECDLP) in G_1 and G_2

* The finite field discrete logarithm problem (FFDLP) in G_T

There are other hard problems over pairing-friendly curves used for proving the security of pairing-based cryptography. Such problems include the computational bilinear Diffie-Hellman (CBDH) problem, the bilinear Diffie-Hellman (BDH) problem, the decision bilinear Diffie-Hellman (DBDH) problem, the gap DBDH problem, etc. [ECRYPT]. Almost all of these variants are reduced to the hardness of discrete logarithm problems described above and are believed to be easier than the discrete logarithm problems.

Although it would be sufficient to attack any of these problems to attack pairing-based crytography, the only known attacks thus far attack the discrete logarithm problem directly, so we focus on the discrete logarithm in this memo.

The security levels of pairing-friendly curves are estimated by the computational cost of the most efficient algorithm for solving the above discrete logarithm problems. The best-known algorithms for solving the discrete logarithm problems are based on Pollard's rho algorithm [Pollard78] and Index Calculus [HR83]. To make index calculus algorithms more efficient, number field sieve (NFS) algorithms are utilized.

3.2. Impact of Recent Attacks

In 2016, Kim and Barbulescu proposed a new variant of the NFS algorithms, the extended tower number field sieve (exTNFS), which drastically reduces the complexity of solving FFDLP [KB16]. The exTNFS improves the polynomial selection that is the first step in the number field sieve algorithm for discrete logarithms in GF(p^k). The idea is applicable when the embedding degree k is a composite that satisfies k = i * j (gcd (i, j) = 1, i, j> 1). Since the above condition is satisfied especially when k = 2^n*3^m (n, m> 1), BN curves and BLS curves whose embedding degree is divisible by 6 are affected by the exTNFS. The basic idea of the exTNFS is based on the equality GF(p^k) = (GF(p^i)^j) and one of the improvement for reducing the amount of cost for solving FFDLP is using sub-field calculation. Please refer to [KB16] for detailed ideas and calculation algorithms of exTNFS. Due to exTNFS, the security levels of certain pairing-friendly curves asymptotically dropped down. For instance, Barbulescu and Duquesne estimated that the security of the BN curves, which had been believed to provide 128-bit security (BN256, for example) was reduced to approximately 100 bits [BD18]. Here, the security levels described in this memo correspond to the security strength of NIST recommendation [NIST].

There has since been research into the minimum bit length of the parameters of pairing-friendly curves for each security level when applying exTNFS as an attacking method for FFDLP. For 128-bit security, Barbulescu and Duquesne estimated the minimum bit length of p of BN curves and BLS12 curves after exTNFS as 461 bits [BD18]. For 256-bit security, Kiyomura et al. estimated the minimum bit length of p^k of BLS48 curves as 27,410 bits, which indicated 572 bits of p [KIK17].

4. Selection of Pairing-Friendly Curves

In this section, we introduce some of the known secure pairing-friendly curves that consider the impact of exTNFS.

First, we show the adoption status of pairing-friendly curves in standards, libraries and applications, and classify them in accordance with the 128-bit, 192-bit, and 256-bit security levels. Then, from the viewpoints of "security" and "widely used", pairing-friendly curves corresponding to each security level are selected and their parameters are indicated.

In our selection policy, it is important that selected curves are shown in peer-reviewed papers for security and that they are widely used in cryptographic libraries. In addition, "efficiency" is one of the important aspects but greatly dependant on implementations, so we choose to prioritize "security" and "widely used" over "efficiency" in consideration of future interconnections and interoperability over the internet.

As a result, we recommend the BLS curve with 381-bit characteristic of embedding degree 12 and the BN curve with the 462-bit characteristic for the 128-bit security level, and the BLS curves of embedding degree 48 with the 581-bit characteristic for the 256-bit security level. On the other hand, we do not show the parameters for 192-bit security here because there are no curves that match our selection policy.

4.1. Adoption Status of Pairing-friendly Curves

We show the pairing-friendly curves that have been selected by existing standards, cryptographic libraries, and applications.

Table 1 summarizes the adoption status of pairing-friendly curves. In this table, "Arnd" is an abbreviation for "Around". The curves categorized as 'Arnd 128-bit', 'Arnd 192-bit' and 'Arnd 256-bit' for each label show that their security levels are within the range of plus/minus 5 bits for each security level. Other labels shown with '~' mean that the security level of the categorized curve is outside the range of each security level. Specifically, the security level of the categorized curves is more than the previous column and is less than the next column. The details are described as the following subsections. A BN curve with a XXX-bit characteristic p is denoted as BNXXX and a BLS curve of embedding degree k with a XXX-bit p is denoted as BLSk_XXX.

Table 1 omits parameters with security levels below the "Arnd 128-bit" range due to space limitations and viewpoints of secure usage of parameters. On the other hand, indicating which standards, libraries, and applications use these lower security level parameters would be useful information for implementers, therefore Appendix D shows these parameters. In addition, the full version of Table 1 is available at https://lepidum.co.jp/blog/2020-03-27/ietf-draft-pfc/.

In Table 1, the security level for each curve is evaluated in accordance with [BD18],[GMT19], [MAF19] and [FK18]. Note that the Freeman curves and MNT curves are not included in this table because [BD18] does not show the security levels of these curves.

Category	**Name**	**Curve Type**	**Security Levels (bit)**				
			Arnd 128	~	Arnd 192	~	Arnd 256
Standard	ISO/IEC	BN384	X				
		BN512I		X			
	TCG	BN638		X			
	FIDO/W3C	BN512I		X			
		BN638		X			
Library	mcl	BLS12_381	X				
		BN382M	X				
		BN462	X				
	RELIC	BLS12_381	X				
		BLS12_446	X				
		BLS12_455	X				
		BLS12_638		X			
		BLS24_477			X		
		BLS48_575					X
		BN382R	X				
		BN446	X				
		BN638		X			
		CP8_544	X				

Category	Name	Curve Type	Security Levels (bit)				
			Arnd 128	~	Arnd 192	~	Arnd 256
		K54_569					X
		KSS18_508		X			
		OT8_511	X				
	AMCL	BLS12_381	X				
		BLS12_383	X				
		BLS12_461	X				
		BLS24_479			X		
		BLS48_556					X
		BN512I		X			
	Kyushu Univ.	BLS48_581					X
	MIRACL	BLS12_381	X				
		BLS12_383	X				
		BLS12_461	X				
		BLS24_479			X		
		BLS48_556					X
		BLS48_581					X
		BN462	X				
		BN512I		X			
	Adjoint	BLS12_381	X				
		BN462	X				
	bls12377js	BLS12_377	X				
Application	Zcash	BLS12_381	X				
	Ethereum	BLS12_381	X				
	Chia Network	BLS12_381	X				
	DFINITY	BLS12_381	X				
		BN382M	X				
		BN462	X				
	Algorand	BLS12_381	X				

Table 1: Adoption Status of Pairing-Friendly Curves

4.1.1. International Standards

ISO/IEC 15946 series specifies public-key cryptographic techniques based on elliptic curves. ISO/IEC 15946-5 [ISOIEC15946-5] shows numerical examples of MNT curves[MNT01] with 160-bit p and 256-bit p, Freeman curves [Freeman06] with 224-bit p and 256-bit p, and BN curves with 160-bit p, 192-bit p, 224-bit p, 256-bit p, 384-bit p, and 512-bit p. These parameters do not take into account the effects of the exTNFS. On the other hand, the parameters may be revised in future versions since ISO/IEC 15946-5 is currently under development. As described below, BN curves with 256-bit p and 512-bit p specified in ISO/IEC 15946-5 used by other standards and libraries, these curves are especially denoted as BN256I and BN512I. The suffix 'I' of BN256I and BN512I are given from the initials of the standard name ISO.

TCG adopts the BN256I and a BN curve with 638-bit p specified by their own[TPM]. FIDO Alliance [FIDO] and W3C [W3C] adopt BN256I, BN512I, the BN638 by TCG, and the BN curve with 256-bit p proposed by Devegili et al.[DSD07] (named BN256D). The suffix 'D' of BN256D is given from the initials of the first author's name of the paper which proposed the parameter.

4.1.2. Cryptographic Libraries

There are a lot of cryptographic libraries that support pairing calculations.

PBC is a library for pairing-based cryptography published by Stanford University that supports BN curves, MNT curves, Freeman curves, and supersingular curves [PBC]. Users can generate pairing parameters by using PBC and use pairing operations with the generated parameters.

mcl[mcl] is a library for pairing-based cryptography that supports four BN curves and BLS12_381 [GMT19]. These BN curves include BN254 proposed by Nogami et al. [NASKM08] (named BN254N), BN_SNARK1 suitable for SNARK applications[libsnark], BN382M, and BN462. The suffix 'N' of BN256N and the suffix 'M' of BN382M are respectively given from the initials of the first author's name of the proposed paper and the library's name mcl. Kyushu University published a library that supports the BLS48_581 [BLS48]. The University of Tsukuba Elliptic Curve and Pairing Library (TEPLA) [TEPLA] supports two BN curves, BN254N and BN254 proposed by Beuchat et al. [BGMORT10] (named BN254B). The suffix 'B' of BN254B is given from the initials of the first author's name of the proposed paper. Intel published a cryptographic library named Intel Integrated Performance Primitives (Intel-IPP) [Intel-IPP] and the library supports BN256I.

RELIC [RELIC] uses various types of pairing-friendly curves including six BN curves (BN158, BN254R, BN256R, BN382R, BN446, and BN638), where BN254R, BN256R, and BN382R are RELIC specific parameters that are different from BN254N, BN254B, BN256I, BN256D, and BN382M. The suffix 'R' of BN382R is given from the initials of the library's name RELIC. In addition, RELIC supports six BLS curves (BLS12_381, BLS12_446, BLS12_445, BLS12_638, BLS24_477, and BLS48_575 [MAF19]), Cocks-Pinch curves of embedding degree 8 with 544-bit p[GMT19], pairing-friendly curves constructed by Scott et al. [SG19] based on Kachisa-Scott-Schaefer curves with embedding degree 54 with 569-bit p (named K54_569)[MAF19], a KSS curve [KSS08] of embedding degree 18 with 508-bit p (named KSS18_508) [AFKMR12], Optimal TNFS-secure curve

[FM19] of embedding degree 8 with 511-bit p(OT8_511), and a supersingular curve [S86] with 1536-bit p (SS_1536).

Apache Milagro Crypto Library (AMCL)[AMCL] supports four BLS curves (BLS12_381, BLS12_461, BLS24_479 and BLS48_556) and four BN curves (BN254N, BN254CX proposed by CertiVox, BN256I, and BN512I). In addition to AMCL's supported curves, MIRACL [MIRACL] supports BN462 and BLS48_581.

Adjoint published a library that supports the BLS12_381 and six BN curves (BN_SNARK1, BN254B, BN254N, BN254S1, BN254S2, and BN462) [AdjointLib], where BN254S1 and BN254S2 are BN curves adopted by an old version of AMCL [AMCLv2]. The suffix 'S' of BN254S1 and BN254S2 are given from the initials of developper's name because he proposed these parameters.

The Celo foundation published the bls12377js library [bls12377js]. The supported curve is the BLS12_377 curve which is shown in [BCGMMW20].

4.1.3. Applications

Zcash uses a BN curve (named BN128) in their library libsnark [libsnark]. In response to the exTNFS attacks, they proposed new parameters using BLS12_381 [BLS12-381] [GMT19]and published its experimental implementation [zkcrypto].

Ethereum 2.0 adopted BLS12_381 and uses the implementation by Meyer [pureGo-bls]. Chia Network published their implementation [Chia] by integrating the RELIC toolkit [RELIC]. DFINITY uses mcl, and Algorand published an implementation which supports BLS12_381.

4.2. For 128-bit Security

Table 1 shows a lot of cases of adopting BN and BLS curves. Among them, BLS12_381 and BN462 match our selection policy. Especially, the one that best matches the policy is BLS12_381 from the viewpoint of "widely used" and "efficiency", so we introduce the parameters of BLS12_381 in this memo.

On the other hand, from the viewpoint of the future use, the parameter of BN462 is also introduced. As shown in recent security evaluations for BLS12_381[BD18] [GMT19], its security level close to 128-bit but it is less than 128-bit. If the attack is improved even a little, BLS12_381 will not be suitable for the curve of the 128-bit security level. As curves of 128-bit security level are currently the most widely used, we recommend both BLS12-381 and BN462 in this memo in order to have a more efficient and a more prudent option respectively.

4.2.1. BLS Curves for the 128-bit security level

In this part, we introduce the parameters of the Barreto-Lynn-Scott curve of embedding degree 12 with 381-bit p that is adopted by a lot of applications such as Zcash [Zcash], Ethereum [Ethereum], and so on.

The BLS12_381 curve is shown in [BLS12-381] and it is defined by the parameter

```
t = -2^63 - 2^62 - 2^60 - 2^57 - 2^48 - 2^16
```

where the size of p becomes 381-bit length.

For the finite field GF(p), the towers of extension field GF(p^2), GF(p^6) and GF(p^12) are defined by indeterminates u, v, and w as follows:

```
GF(p^2) = GF(p)[u] / (u^2 + 1)
GF(p^6) = GF(p^2)[v] / (v^3 - u - 1)
GF(p^12) = GF(p^6)[w] / (w^2 - v).
```

Defined by t, the elliptic curve E and its twist E' are represented by E: y^2 = x^3 + 4 and E': y^2 = x^3 + 4(u + 1). BLS12_381 is categorized as M-type.

We have to note that the security level of this pairing is expected to be 126 rather than 128 bits [GMT19].

Parameters of BLS12_381 are given as follows.

- *G_1 is the largest prime-order subgroup of E(GF(p))
  - -BP = (x,y) : a 'base point', i.e., a generator of G_1
- *G_2 is an r-order subgroup of E'(GF(p^2))
  - -BP' = (x',y') : a 'base point', i.e., a generator of G_2 (encoded with [I-D.ietf-lwig-curve-representations])
    - ox' = x'_0 + x'_1 * u (x'_0, x'_1 in GF(p))
    - oy' = y'_0 + y'_1 * u (y'_0, y'_1 in GF(p))
  - -h' : the cofactor #E'(GF(p^2))/r

p:
0x1a0111ea397fe69a4b1ba7b6434bacd764774b84f38512bf6730d2a0f6b0f6241eabfffeb153ffffb9fef

r:
0x73eda753299d7d483339d80809a1d80553bda402fffe5bfeffffffff00000001

x:
0x17f1d3a73197d7942695638c4fa9ac0fc3688c4f9774b905a14e3a3f171bac586c55e83ff97a1aeffb3af

y:
0x08b3f481e3aaa0f1a09e30ed741d8ae4fcf5e095d5d00af600db18cb2c04b3edd03cc744a2888ae40caa2

h: 0x396c8c005555e1568c00aaab0000aaab

b: 4

x'_0:
0x024aa2b2f08f0a91260805272dc51051c6e47ad4fa403b02b4510b647ae3d1770bac0326a805bbefd4805

x'_1:
0x13e02b6052719f607dacd3a088274f65596bd0d09920b61ab5da61bbdc7f5049334cf11213945d57e5ac7

y'_0:
0x0ce5d527727d6e118cc9cdc6da2e351aadfd9baa8cbdd3a76d429a695160d12c923ac9cc3baca289e1935

y'_1:
0x0606c4a02ea734cc32acd2b02bc28b99cb3e287e85a763af267492ab572e99ab3f370d275cec1da1aaa9

h':
0x5d543a95414e7f1091d50792876a202cd91de4547085abaa68a205b2e5a7ddfa628f1cb4d9e82ef21537e

b': 4 * (u + 1)

As mentioned above, BLS12_381 is adopted in a lot of applications. Since it is expected that BLS12_381 will continue to be widely used more and more in the future, Appendix C shows the serialization format of points on an elliptic curve as useful information. This serialization format is also adopted in [I-D.boneh-bls-signature] [zkcrypto].

In addition, many pairing-based cryptographic applications use a hashing to an elliptic curve procedure that outputs a rational point on an elliptic curve from an arbitrary input. A standard specification of ciphersuites for a hashing to an elliptic curve, including BLS12-381, is under discussion in the IETF [I-D.irtf-cfrg-hash-to-curve] and it will be valuable information for implementers.

4.2.2. BN Curves for the 128-bit security level

A BN curve with the 128-bit security level is shown in [BD18], which we call BN462. BN462 is defined by the parameter

```
t = 2^114 + 2^101 - 2^14 - 1
```

for the definition in Section 2.3.

For the finite field GF(p), the towers of extension field GF(p^2), GF(p^6) and GF(p^12) are defined by indeterminates u, v, and w as follows:

```
GF(p^2) = GF(p)[u] / (u^2 + 1)
GF(p^6) = GF(p^2)[v] / (v^3 - u - 2)
GF(p^12) = GF(p^6)[w] / (w^2 - v).
```

Defined by t, the elliptic curve E and its twist E' are represented by E: y^2 = x^3 + 5 and E': y^2 = x^3 - u + 2, respectively. The size of p becomes 462-bit length. BN462 is categorized as D-type.

We have to note that BN462 is significantly slower than BLS12_381, but has 134-bit security level [GMT19], so may be more resistant to future small improvements to the exTNFS attack.

We note also that CP8_544 is about 20% faster that BN462 [GMT19], has 131-bit security level, and that due to its construction will not be affected by future small improvements to the exTNFS attack. However, as this curve is not widely used (it is only implemented in one library), we instead chose BN462 for our 'safe' option.

We give the following parameters for BN462.

- *G_1 is the largest prime-order subgroup of E(GF(p))

  - -BP = (x,y) : a 'base point', i.e., a generator of G_1

- *G_2 is an r-order subgroup of E'(GF(p^2))

  - -BP' = (x',y') : a 'base point', i.e., a generator of G_2 (encoded with [I-D.ietf-lwig-curve-representations])

    - ox' = x'_0 + x'_1 * u (x'_0, x'_1 in GF(p))

    - oy' = y'_0 + y'_1 * u (y'_0, y'_1 in GF(p))

  - -h' : the cofactor #E'(GF(p^2))/r

p:

0x240480360120023ffffffffff6ff0cf6b7d9bfca0000000000d812908f41c8020ffffffffff6ff66fc6f

r:
0x240480360120023ffffffffff6ff0cf6b7d9bfca0000000000d812908ee1c201f7fffffffff6ff66fc7bf

x:
0x21a6d67ef250191fadba34a0a30160b9ac9264b6f95f63b3edbec3cf4b2e689db1bbb4e69a416a0b1e792

y:
0x0118ea0460f7f7abb82b33676a7432a490eeda842cccfa7d788c659650426e6af77df11b8ae40eb80f475

h: 1

b: 5

x'_0:
0x0257ccc85b58dda0dfb38e3a8cbdc5482e0337e7c1cd96ed61c913820408208f9ad2699bad92e0032ae1f

x'_1:
0x1d2e4343e8599102af8edca849566ba3c98e2a354730cbed9176884058b18134dd86bae555b783718f50a

y'_0:
0x0a0650439da22c1979517427a20809eca035634706e23c3fa7a6bb42fe810f1399a1f41c9ddae32e03695

y'_1:
0x073ef0cbd438cbe0172c8ae37306324d44d5e6b0c69ac57b393f1ab370fd725cc647692444a04ef87387a

h':
0x240480360120023ffffffffff6ff0cf6b7d9bfca0000000000d812908fa1ce0227fffffffff6ff66fc63f

b': -u + 2

4.3. For 256-bit Security

As shown in Table 1, there are three candidates of pairing-friendly curves for 256-bit security. According to our selection policy, we select BLS48_581, as it is the most widely adopted by cryptographic libraries.

The selected BLS48 curve is shown in [KIK17] and it is defined by the parameter

```
t = -1 + 2^7 - 2^10 - 2^30 - 2^32.
```

In this case, the size of p becomes 581-bit.

For the finite field GF(p), the towers of extension field GF(p^2), GF(p^4), GF(p^8), GF(p^24) and GF(p^48) are defined by indeterminates u, v, w, z, and s as follows:

```
GF(p^2) = GF(p)[u] / (u^2 + 1)
GF(p^4) = GF(p^2)[v] / (v^2 + u + 1)
GF(p^8) = GF(p^4)[w] / (w^2 + v)
GF(p^24) = GF(p^8)[z] / (z^3 + w)
GF(p^48)= GF(p^24)[s] / (s^2 + z).
```

The elliptic curve E and its twist E' are represented by E: y^2 = x^3 + 1 and E': y^2 = x^3 - 1 / w. BLS48-581 is categorized as D-type.

We then give the parameters for BLS48-581 as follows.

- *G_1 is the largest prime-order subgroup of E(GF(p))
  - -BP = (x,y) : a 'base point', i.e., a generator of G_1
- *G_2 is an r-order subgroup of E'(GF(p^8))
  - -BP' = (x',y') : a 'base point', i.e., a generator of G_2 (encoded with [I-D.ietf-lwig-curve-representations])

    ox' = x'_0 + x'_1 * u + x'_2 * v + x'_3 * u * v + x'_4 * w + x'_5 * u * w + x'_6 * v * w + x'_7 * u * v * w (x'_0, ..., x'_7 in GF(p))

    oy' = y'_0 + y'_1 * u + y'_2 * v + y'_3 * u * v + y'_4 * w + y'_5 * u * w + y'_6 * v * w + y'_7 * u * v * w (y'_0, ..., y'_7 in GF(p))
  - -h' : the cofactor #E'(GF(p^8))/r

p:
0x1280f73ff3476f313824e31d47012a0056e84f8d122131bb3be6c0f1f3975444a48ae43af6e082acd9cd3

r:
0x2386f8a925e2885e233a9ccc1615c0d6c635387a3f0b3cbe003fad6bc972c2e6e741969d34c4c92016a85

x:
0x02af59b7ac340f2baf2b73df1e93f860de3f257e0e86868cf61abdbaedffb9f7544550546a9df6f964584

y:

0x0cefda44f6531f91f86b3a2d1fb398a488a553c9efeb8a52e991279dd41b720ef7bb7beffb98aee53e80f

x'_0:

0x05d615d9a7871e4a38237fa45a2775debabbefc70344dbccb7de64db3a2ef156c46ff79baad1a8c42281a

x'_1:

0x07c4973ece2258512069b0e86abc07e8b22bb6d980e1623e9526f6da12307f4e1c3943a00abfedf16214a

x'_2:

0x01fccc70198f1334e1b2ea1853ad83bc73a8a6ca9ae237ca7a6d6957ccbab5ab6860161c1dbd19242ffae

x'_3:

0x0be2218c25ceb6185c78d8012954d4bfe8f5985ac62f3e5821b7b92a393f8be0cc218a95f63e1c776e6ec

x'_4:

0x038b91c600b35913a3c598e4caa9dd63007c675d0b1642b5675ff0e7c5805386699981f9e48199d5ac10b

x'_5:

0x0c96c7797eb0738603f1311e4ecda088f7b8f35dcef0977a3d1a58677bb037418181df63835d28997eb57

x'_6:

0x0b9b7951c6061ee3f0197a498908aee660dea41b39d13852b6db908ba2c0b7a449cef11f293b13ced0fd0

x'_7:

0x0827d5c22fb2bdec5282624c4f4aaa2b1e5d7a9defaf47b5211cf741719728a7f9f8cfca93f29cff364a7

y'_0:

0x00eb53356c375b5dfa497216452f3024b918b4238059a577e6f3b39ebfc435faab0906235afa27748d90f

y'_1:

0x0284dc75979e0ff144da6531815fcadc2b75a422ba325e6fba01d72964732fcbf3afb096b243b1f192c5c

y'_2:

0x0b36a201dd008523e421efb70367669ef2c2fc5030216d5b119d3a480d370514475f7d5c99d0e90411515

y'_3:

0x0aec25a4621edc0688223fbbd478762b1c2cded3360dcee23dd8b0e710e122d2742c89b224333fa40dced

y'_4:
0x0d209d5a223a9c46916503fa5a88325a2554dc541b43dd93b5a959805f1129857ed85c77fa238cdce8a1e

y'_5:
0x07d0d03745736b7a513d339d5ad537b90421ad66eb16722b589d82e2055ab7504fa83420e8c270841f682

y'_6:
0x0896767811be65ea25c2d05dfdd17af8a006f364fc0841b064155f14e4c819a6df98f425ae3a2864f22c1

y'_7:
0x035e2524ff89029d393a5c07e84f981b5e068f1406be8e50c87549b6ef8eca9a9533a3f8e69c31e97e1ad

h: 0x85555841aaaec4ac

b: 1

h':
0x170e915cb0a6b7406b8d94042317f811d6bc3fc6e211ada42e58ccfcb3ac076a7e4499d700a0c23dc4b0c

b': -1 / w

5. Security Considerations

The recommended pairing-friendly curves are selected by considering the exTNFS proposed by Kim et al. in 2016 [KB16] and they are categorized in each security level in accordance with [BD18]. Implementers who will newly develop pairing-based cryptography applications SHOULD use the recommended parameters. As of 2020, as far as we've investigated the top cryptographic conferences in the past, there are no fatal attacks that significantly reduce the security of pairing-friendly curves after exTNFS.

BLS curves of embedding degree 12 typically require a characteristic p of 461 bits or larger to achieve the 128-bit security level [BD18]. Note that the security level of BLS12-381, which is adopted by a lot of libraries and applications, is slightly below 128 bits because a 381-bit characteristic is used [BD18] [GMT19].

BN254 is used in most of the existing implementations as shown in Section 4.1 (and Appendix D), however, BN curves that were estimated as the 128-bit security level before exTNFS including BN254 ensure no more than the 100-bit security level by the effect of exTNFS.

In addition, implementors should be aware of the following points when they implement pairing-based cryptographic applications using recommended curves. Regarding the use case and applications of pairing-based cryptographic applications, please refer Section 1.2.

In applications such as key agreement protocols, users exchange the elements in G_1 and G_2 as public keys. To check these elements are so-called sub-group secure [BCM15], implementors should validate if the elements have the correct order r. Specifically, for public keys P in G_1 and Q in G_2, a receiver should calculate scalar multiplications [r]P and [r]Q, and check the results become points at infinity.

The pairing-based protocols, such as the BLS signatures, use a scalar multiplication in G_1, G_2 and an exponentiation in G_3 with the secret key. In order to prevent the leakage of secret key due to side channel attacks, implementors should apply countermeasure techniques such as montgomery ladder [Montgomery] [CF06] when they implement modules of a scalar multiplication and an exponentiation. Please refer [Montgomery] and [CF06] for the detailed algorithms of montgomery ladder.

When converting between an element in extension field and an octet string, implementors should check that the coefficient is within an appropriate range [IEEE1363]. If the coefficient is out of range, there is a possible that security vulnerabilities such as the signature forgery may occur.

Recommended parameters are affected by the Cheon's attack which is a solving algorithm for the strong DH problem [Cheon06]. The mathematical problem that provides the security of the strong DH problem is called ECDLP with Auxiliary Inputs (ECDLPwAI). In ECDLPwAI, given rational points P, [K]P, [K^i]P, for i=1,...,n, then we find a secret K. Since the complexity of ECDLPwAI is given as O(sqrt((r-1)/n + sqrt(n)) where n|r-1 by using Cheon's algorithm whereas the complexity of ECDLP is given as O(sqrt(r)), the complexity of ECDLPwAI with the ideal value n becomes dramatically smaller than that of ECDLP. Please refer [Cheon06] for the details of Cheon's algorithm. Therefore, implementers should be careful when they design cryptographic protocols based on the strong DH problem. For example, in the case of Short Signatures, they can prevent the Cheon's attack by carefully setting the maximum number of queries which corresponds to the parameter n.

6. IANA Considerations

This document has no actions for IANA.

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Appendix A. Computing the Optimal Ate Pairing

Before presenting the computation of the optimal Ate pairing e(P, Q) satisfying the properties shown in Section 2.2, we give the subfunctions used for the pairing computation.

The following algorithm, Line_Function shows the computation of the line function. It takes A = (A[1], A[2]), B = (B[1], B[2]) in G_2, and P = ((P[1], P[2])) in G_1 as input, and outputs an element of G_T.

```
if (A = B) then
    l := (3 * A[1]^2) / (2 * A[2]);
else if (A = -B) then
    return P[1] - A[1];
else
    l := (B[2] - A[2]) / (B[1] - A[1]);
end if;
return (l * (P[1] -A[1]) + A[2] -P[2]);
```

When implementing the line function, implementers should consider the isomorphism of E and its twist curve E' so that one can reduce the computational cost of operations in G_2 [CLN09][KIK17]. We note that Line_function does not consider such an isomorphism.

The computation of the optimal Ate pairing for BN curves uses the Frobenius map. The p-power Frobenius map pi for a point Q = (x, y) over E' is pi(p, Q) = (x^p, y^p).

A.1. Optimal Ate Pairings over Barreto-Naehrig Curves

Let c = 6 * t + 2 for a parameter t and c_0, c_1, ... , c_L in {-1,0,1} such that the sum of c_i * 2^i (i = 0, 1, ..., L) equals c.

The following algorithm shows the computation of the optimal Ate pairing on BN curves. It takes P in G_1, Q in G_2, an integer c, c_0, ...,c_L in {-1,0,1} such that the sum of c_i * 2^i (i = 0, 1, ..., L) equals c, and the order r of G_1 as input, and outputs e(P, Q).

```
f := 1; T := Q;
if (c_L = -1)
    T := -T;
end if
for i = L-1 to 0
    f := f^2 * Line_function(T, T, P); T := 2 * T;
    if (c_i = 1 | c_i = -1)
        f := f * Line_function(T, c_i * Q); T := T + c_i * Q;
    end if
end for
Q_1 := pi(p, Q); Q_2 := pi(p, Q_1);
f := f * Line_function(T, Q_1, P); T := T + Q_1;
f := f * Line_function(T, -Q_2, P);
f := f^{(p^k - 1) / r}
return f;
```

A.2. Optimal Ate Pairings over Barreto-Lynn-Scott Curves

Let c = t for a parameter t and c_0, c_1, ... , c_L in {-1,0,1} such that the sum of c_i * 2^i (i = 0, 1, ..., L) equals c. The following algorithm shows the computation of optimal Ate pairing over Barreto-Lynn-Scott curves. It takes P in G_1, Q in G_2, a parameter c, c_0, c_1, ..., c_L in {-1,0,1} such that the sum of c_i * 2^i (i = 0, 1, ..., L), and an order r as input, and outputs e(P, Q).

```
f := 1; T := Q;
if (c_L = -1)
    T := -T;
end if
for i = L-1 to 0
    f := f^2 * Line_function(T, T, P); T := 2 * T;
    if (c_i = 1 | c_i = -1)
        f := f * Line_function(T, c_i * Q, P); T := T + c_i * Q;
    end if
end for
f := f^{(p^k - 1) / r};
return f;
```

Appendix B. Test Vectors of Optimal Ate Pairing

We provide test vectors for Optimal Ate Pairing e(P, Q) given in Appendix A for the curves BLS12-381, BN462 and BLS48-581 given in Section 4. Here, the inputs P = (x, y) and Q = (x', y') are the corresponding base points BP and BP' given in Section 4.

For BLS12-381 and BN462, Q = (x', y') is given by

```
x' = x'_0 + x'_1 * u and
y' = y'_0 + y'_1 * u,
```

where u is a indeterminate and x'_0, x'_1, y'_0, y'_1 are elements of GF(p).

For BLS48-581, Q = (x', y') is given by

```
x' = x'_0 + x'_1 * u + x'_2 * v + x'_3 * u * v
    + x'_4 * w + x'_5 * u * w + x'_6 * v * w + x'_7 * u * v * w and
y' = y'_0 + y'_1 * u + y'_2 * v + y'_3 * u * v
    + y'_4 * w + y'_5 * u * w + y'_6 * v * w + y'_7 * u * v * w,
```

where u, v and w are indeterminates and x'_0, ..., x'_7 and y'_0, ..., y'_7 are elements of GF(p). The representation of Q = (x', y') given below is followed by [I-D.ietf-lwig-curve-representations].

BLS12-381:

Input x value:

0x17f1d3a73197d7942695638c4fa9ac0fc3688c4f9774b905a14e3a3f171bac586c55e83ff97a1aeffb3af

Input y value:

0x08b3f481e3aaa0f1a09e30ed741d8ae4fcf5e095d5d00af600db18cb2c04b3edd03cc744a2888ae40caa2

Input x'_0 value:

0x024aa2b2f08f0a91260805272dc51051c6e47ad4fa403b02b4510b647ae3d1770bac0326a805bbefd4805

Input x'_1 value:

0x13e02b6052719f607dacd3a088274f65596bd0d09920b61ab5da61bbdc7f5049334cf11213945d57e5ac7

Input y'_0 value:

0x0ce5d527727d6e118cc9cdc6da2e351aadfd9baa8cbdd3a76d429a695160d12c923ac9cc3baca289e1935

Input y'_1 value:

0x0606c4a02ea734cc32acd2b02bc28b99cb3e287e85a763af267492ab572e99ab3f370d275cec1da1aaa90

e_0:

0x11619b45f61edfe3b47a15fac19442526ff489dcda25e59121d9931438907dfd448299a87dde3a649bdba

e_1:

0x153ce14a76a53e205ba8f275ef1137c56a566f638b52d34ba3bf3bf22f277d70f76316218c0dfd583a394

e_2:

0x095668fb4a02fe930ed44767834c915b283b1c6ca98c047bd4c272e9ac3f3ba6ff0b05a93e59c71fba77b

e_3:

0x16deedaa683124fe7260085184d88f7d036b86f53bb5b7f1fc5e248814782065413e7d958d17960109ea0

e_4:

0x09c92cf02f3cd3d2f9d34bc44eee0dd50314ed44ca5d30ce6a9ec0539be7a86b121edc61839ccc908c4b

e_5:

0x111061f398efc2a97ff825b04d21089e24fd8b93a47e41e60eae7e9b2a38d54fa4dedced0811c34ce5287

e_6:

0x01ecfcf31c86257ab00b4709c33f1c9c4e007659dd5ffc4a735192167ce197058cfb4c94225e7f1b6c26a

e_7:
0x08890726743a1f94a8193a166800b7787744a8ad8e2f9365db76863e894b7a11d83f90d873567e9d645cc

e_8:
0x0e61c752414ca5dfd258e9606bac08daec29b3e2c57062669556954fb227d3f1260eedf25446a086b0844

e_9:
0x0fe63f185f56dd29150fc498bbeea78969e7e783043620db33f75a05a0a2ce5c442beaff9da195ff15164

e_10:
0x10900338a92ed0b47af211636f7cfdec717b7ee43900eee9b5fc24f0000c5874d4801372db478987691c5

e_11:
0x1454814f3085f0e6602247671bc408bbce2007201536818c901dbd4d2095dd86c1ec8b888e59611f60a30

BN462:

Input x value:

    0x21a6d67ef250191fadba34a0a30160b9ac9264b6f95f63b3edbec3cf4b2e689db1bbb4e69a416a0b1e79

Input y value:

    0x0118ea0460f7f7abb82b33676a7432a490eeda842cccfa7d788c659650426e6af77df11b8ae40eb80f47

Input x'_0 value:

    0x0257ccc85b58dda0dfb38e3a8cbdc5482e0337e7c1cd96ed61c913820408208f9ad2699bad92e0032ae1

Input x'_1 value:

    0x1d2e4343e8599102af8edca849566ba3c98e2a354730cbed9176884058b18134dd86bae555b783718f50

Input y'_0 value:

    0x0a0650439da22c1979517427a20809eca035634706e23c3fa7a6bb42fe810f1399a1f41c9ddae32e0369

Input y'_1 value:

    0x073ef0cbd438cbe0172c8ae37306324d44d5e6b0c69ac57b393f1ab370fd725cc647692444a04ef87387

e_0:

    0x0cf7f0f2e01610804272f4a7a24014ac085543d787c8f8bf07059f93f87ba7e2a4ac77835d4ff10e7866

e_1:

    0x00ef2c737515694ee5b85051e39970f24e27ca278847c7cfa709b0df408b830b3763b1b001f1194445b6

e_2:

    0x04d685b29fd2b8faedacd36873f24a06158742bb2328740f93827934592d6f1723e0772bb9ccd3025f88

e_3:

    0x090067ef2892de0c48ee49cbe4ff1f835286c700c8d191574cb424019de11142b3c722cc5083a7191241

e_4:

    0x1437603b60dce235a090c43f5147d9c03bd63081c8bb1ffa7d8a2c31d673230860bb3dfe4ca85581f745

e_5:

    0x13191b1110d13650bf8e76b356fe776eb9d7a03fe33f82e3fe5732071f305d201843238cc96fd0e892bc

e_6:

0x07b1ce375c0191c786bb184cc9c08a6ae5a569dd7586f75d6d2de2b2f075787ee5082d44ca4b8009b3285

e_7:

0x05b64add5e49574b124a02d85f508c8d2d37993ae4c370a9cda89a100cdb5e1d441b57768dbc68429ffae

e_8:

0x0fd9a3271854a2b4542b42c55916e1faf7a8b87a7d10907179ac7073f6a1de044906ffaf4760d11c8f92c

e_9:

0x17fa0c7fa60c9a6d4d8bb9897991efd087899edc776f33743db921a689720c82257ee3c788e8160c112f1

e_10:

0x0c901397a62bb185a8f9cf336e28cfb0f354e2313f99c538cdceedf8b8aa22c23b896201170fc915690f7

e_11:

0x20f27fde93cee94ca4bf9ded1b1378c1b0d80439eeb1d0c8daef30db0037104a5e32a2ccc94fa1860a95e

BLS48-581:

Input x value:

0x02af59b7ac340f2baf2b73df1e93f860de3f257e0e86868cf61abdbaedffb9f7544550546a9df6f964584

Input y value:

0x0cefda44f6531f91f86b3a2d1fb398a488a553c9efeb8a52e991279dd41b720ef7bb7beffb98aee53e80f

x'_0:

0x05d615d9a7871e4a38237fa45a2775debabbefc70344dbccb7de64db3a2ef156c46ff79baad1a8c42281a

x'_1:

0x07c4973ece2258512069b0e86abc07e8b22bb6d980e1623e9526f6da12307f4e1c3943a00abfedf16214a

x'_2:

0x01fccc70198f1334e1b2ea1853ad83bc73a8a6ca9ae237ca7a6d6957ccbab5ab6860161c1dbd19242ffae

x'_3:

0x0be2218c25ceb6185c78d8012954d4bfe8f5985ac62f3e5821b7b92a393f8be0cc218a95f63e1c776e6ec

x'_4:

0x038b91c600b35913a3c598e4caa9dd63007c675d0b1642b5675ff0e7c5805386699981f9e48199d5ac10b

x'_5:

0x0c96c7797eb0738603f1311e4ecda088f7b8f35dcef0977a3d1a58677bb037418181df63835d28997eb57

x'_6:

0x0b9b7951c6061ee3f0197a498908aee660dea41b39d13852b6db908ba2c0b7a449cef11f293b13ced0fd0

x'_7:

0x0827d5c22fb2bdec5282624c4f4aaa2b1e5d7a9defaf47b5211cf741719728a7f9f8cfca93f29cff364a7

y'_0:

0x00eb53356c375b5dfa497216452f3024b918b4238059a577e6f3b39ebfc435faab0906235afa27748d90f

y'_1:

0x0284dc75979e0ff144da6531815fcadc2b75a422ba325e6fba01d72964732fcbf3afb096b243b1f192c5c

y'_2:

0x0b36a201dd008523e421efb70367669ef2c2fc5030216d5b119d3a480d370514475f7d5c99d0e90411515

y'_3:
0x0aec25a4621edc0688223fbbd478762b1c2cded3360dcee23dd8b0e710e122d2742c89b224333fa40dced

y'_4:
0x0d209d5a223a9c46916503fa5a88325a2554dc541b43dd93b5a959805f1129857ed85c77fa238cdce8a1e

y'_5:
0x07d0d03745736b7a513d339d5ad537b90421ad66eb16722b589d82e2055ab7504fa83420e8c270841f682

y'_6:
0x0896767811be65ea25c2d05dfdd17af8a006f364fc0841b064155f14e4c819a6df98f425ae3a2864f22c1

y'_7:
0x035e2524ff89029d393a5c07e84f981b5e068f1406be8e50c87549b6ef8eca9a9533a3f8e69c31e97e1ac

e_0:
0x0e26c3fcb8ef67417814098de5111ffcccc1d003d15b367bad07cef2291a93d31db03e3f03376f3beae2b

e_1:
0x069061b8047279aa5c2d25cdf676ddf34eddbc8ec2ec0f03614886fa828e1fc066b26d35744c0c3827184

e_2:
0x02b9bece645fbf9d8f97025a1545359f6fe3ffab3cd57094f862f7fb9ca01c88705c26675bcc723878e94

e_3:
0x0080d267bf036c1e61d7fc73905e8c630b97aa05ef3266c82e7a111072c0d2056baa8137fba111c9650df

e_4:
0x03c6b4c12f338f9401e6a493a405b33e64389338db8c5e592a8dd79eac7720dd83dd6b0c189eeda208091

e_5:
0x016e46224f28bfd8833f76ac29ee6e406a9da1bde55f5e82b3bd977897a9104f18b9ee41ea9af7d4183d8

e_6:
0x008ddce7a4a1b94be5df3ceea56bef0077dcdde86d579938a50933a47296d337b7629934128e2457e2414

e_7:

0x060ef6eae55728e40bd4628265218b24b38cdd434968c14bfefb87f0dcbfc76cc473ae2dc0cac6e69dfd

e_8:

0x0c3943636876fd4f9393414099a746f84b2633dfb7c36ba6512a0b48e66dcb2e409f1b9e150e36b0b431

e_9:

0x02d31eb8be0d923cac2a8eb6a07556c8951d849ec53c2848ee78c5eed40262eb21822527a8555b071f1c

e_10:

0x07f19673c5580d6a10d09a032397c5d425c3a99ff1dd0abe5bec40a0d47a6b8daabb22edb6b06dd86919

e_11:

0x0d3fe01f0c114915c3bdf8089377780076c1685302279fd9ab12d07477aac03b69291652e9f179baa0a9

e_12:

0x0662eefd5fab9509aed968866b68cff3bc5d48ecc8ac6867c212a2d82cee5a689a3c9c67f1d611adac72

e_13:

0x0aad8f4a8cfdca8de0985070304fe4f4d32f99b01d4ea50d9f7cd2abdc0aeea99311a36ec6ed18208642

e_14:

0x0ffcf21d641fd9c6a641a749d80cab1bcad4b34ee97567d905ed9d5cfb74e9aef19674e2eb6ce3dfb706

e_15:

0x0cbe92a53151790cece4a86f91e9b31644a86fc4c954e5fa04e707beb69fc60a858fed8ebd53e4cfd515

e_16:

0x0202db83b1ff33016679b6cfc8931deea6df1485c894dcd113bacf564411519a42026b5fda4e16262674

e_17:

0x070a617ed131b857f5b74b625c4ef70cc567f619defb5f2ab67534a1a8aa72975fc4248ac8551ce02b68

e_18:

0x070e1ebce457c141417f88423127b7a7321424f64119d5089d883cb953283ee4e1f2e01ffa7b903fe7a9

e_19:

0x058a06be5a36c6148d8a1287ee7f0e725453fa1bb05cf77239f235b417127e370cfa4f88e61a23ea16df3

e_20:
0x0dfdfaaeb9349cf18d21b92ad68f8a7ecc509c35fcd4b8abeb93be7a204ac871f2195180206a2c340fccb

e_21:
0x0d06c8adfdd81275da2a0ce375b8df9199f3d359e8cf50064a3dc10a592417124a3b705b05a7ffe78e20f

e_22:
0x0708effd28c4ae21b6969cb9bdd0c27f8a3e341798b6f6d4baf27be259b4a47688b50cb68a69a917a4a1f

e_23:
0x09da7c7aa48ce571f8ece74b98431b14ae6fb4a53ae979cd6b2e82320e8d25a0ece1ca1563aa5aa6926e7

e_24:
0x0a7150a14471994833d89f41daeaa999dfc24a9968d4e33d88ed9e9f07aa2432c53e486ba6e3b6e4f4b8d

e_25:
0x084696f31ff27889d4dccdc4967964a5387a5ae071ad391c5723c9034f16c2557915ada07ec68f18672b5

e_26:
0x0398e76e3d2202f999ac0f73e0099fe4e0fe2de9d223e78fc65c56e209cdf48f0d1ad8f6093e924ce5f0c

e_27:
0x06d683f556022368e7a633dc6fe319fd1d4fc0e07acff7c4d4177e83a911e73313e0ed980cd9197bd17ac

e_28:
0x0d764075344b70818f91b13ee445fd8c1587d1c0664002180bbac9a396ad4a8dc1e695b0c4267df4a0908

e_29:
0x0aa6a32fdc4423b1c6d43e5104159bcd8e03a676d055d4496f7b1bc8761164a2908a3ff0e4c4d1f43620

e_30:
0x1147719959ac8eeab3fc913539784f1f947df47066b6c0c1beafecdb5fa784c3be9de5ab282a678a2a0cb

e_31:
0x11a377bcebd3c12702bb34044f06f8870ca712fb5caa6d30c48ace96898fcbcddbcf31f331c9e524684c0

e_32:
0x0b8b4511f451ba2cc58dc28e56d5e1d0a8f557ecb242f4d994a627e07cf3fa44e6d83cb907deacf303d2f

e_33:
0x090962d632ee2a57ce4208052ce47a9f76ea0fdad724b7256bb07f3944e9639a981d3431087241e30ae9b

e_34:
0x0931c7befc80acd185491c68af886fa8ee39c21ed3ebd743b9168ae3b298df485bfdc75b94f0b21aecd8c

e_35:
0x020ac007bf6c76ec827d53647058aca48896916269c6a2016b8c06f0130901c8975779f1672e581e2dfdb

e_36:
0x0c0aed0d890c3b0b673bf4981398dcbf0d15d36af6347a39599f3a22584184828f78f91bbbbd08124a976

e_37:
0x0ef7799241a1ba6baaa8740d5667a1ace50fb8e63accc3bc30dc07b11d78dc545b68910c027489a0d842c

e_38:
0x016663c940d062f4057257c8f4fb9b35e82541717a34582dd7d55b41ebadf40d486ed74570043b2a3c4de

e_39:
0x1184a79510edf25e3bd2dc793a5082fa0fed0d559fa14a5ce9ffca4c61f17196e1ffbb84326272e0d0793

e_40:
0x120e47a747d942a593d202707c936dafa6fed489967dd94e48f317fd3c881b1041e3b6bbf9e8031d44e39

e_41:
0x026b6e374108ecb2fe8d557087f40ab7bac8c5af0644a655271765d57ad71742aa331326d871610a8c4c3

e_42:
0x041be63a2fa643e5a66faeb099a3440105c18dca58d51f74b3bf281da4e689b13f365273a2ed397e7b1c2

e_43:
0x124018a12f0f0af881e6765e9e81071acc56ebcddadcd107750bd8697440cc16f190a3595633bb8900e68

e_44:

0x0d422de4a83449c535b4b9ece586754c941548f15d50ada6740865be9c0b066788b6078727c7dee299ac

e_45:
0x1119f6c5468bce2ec2b450858dc073fea4fb05b6e83dd20c55c9cf694cbcc57fc0effb1d33b9b5587852

e_46:
0x061eaa8e9b0085364a61ea4f69c3516b6bf9f79f8c79d053e646ea637215cf6590203b275290872e3d7b2

e_47:
0x0add8d58e9ec0c9393eb8c4bc0b08174a6b421e15040ef558da58d241e5f906ad6ca2aa5de361421708a6

Appendix C. ZCash serialization format for BLS12-381

This section describes the serialization format defined by [ZCashRep]. This format applies to points on the BLS12-381 elliptic curves E and E', whose parameters are given in Section 4.2.1. Note that this serialization method is based on the representation shown in [SEC1] and it is a tiny tweak so as to apply to GF(p^m). It is not officially standardized by the standards organization, however we show it in this appendix as a useful reference for implementers.

At a high level, the serialization format is defined as follows:

* Serialized points include three metadata bits that indicate whether a point is compressed or not, whether a point is the point at infinity or not, and (for compressed points) the sign of the point's y-coordinate.

* Points on E are serialized into 48 bytes (compressed) or 96 bytes (uncompressed). Points on E' are serialized into 96 bytes (compressed) or 192 bytes (uncompressed).

* The serialization of a point at infinity comprises a string of zero bytes, except that the metadata bits may be nonzero.

* The serialization of a compressed point other than the point at infinity comprises a serialized x-coordinate.

* The serialization of an uncompressed point other than the point at infinity comprises a serialized x-coordinate followed by a serialized y-coordinate.

Below, we give detailed serialization and de-serialization procedures. The following notation is used in the rest of this section:

* Elements of GF(p^2) are represented as polynomial with GF(p) coefficients like Section 2.5.

* For a byte string str, str[0] is defined as the first byte of str.

* The function sign_GF_p(y) returns one bit representing the sign of an element of GF(p). This function is defined as follows:

```
sign_GF_p(y) := { 1 if y > (p - 1) / 2, else
                { 0 otherwise.
```

* The function sign_GF_p^2(y') returns one bit representing the sign of an element in GF(p^2). This function is defined as follows:

```
sign_GF_p^2(y') := { sign_GF_p(y'_0) if y'_1 equals 0, else
                   { 1 if y'_1 > (p - 1) / 2, else
                   { 0 otherwise.
```

C.1. Point Serialization Procedure

The serialization procedure is defined as follows for a point P = (x, y). This procedure uses the I2OSP function defined in [RFC8017].

1. Compute the metadata bits C_bit, I_bit, and S_bit, as follows:

   * C_bit is 1 if point compression should be used, otherwise it is 0.

   * I_bit is 1 if P is the point at infinity, otherwise it is 0.

   * S_bit is 0 if P is the point at infinity or if point compression is not used. Otherwise (i.e., when point compression is used and P is not the point at infinity), if P is a point on E, S_bit = sign_GF_p(y), else if P is a point on E', S_bit = sign_GF_p^2(y).

2. Let m_byte = (C_bit * 2^7) + (I_bit * 2^6) + (S_bit * 2^5).

3. Let x_string be the serialization of x, which is defined as follows:

   * If P is the point at infinity on E, let x_string = I2OSP(0, 48).

*If P is a point on E other than the point at infinity, then x is an element of GF(p), i.e., an integer in the inclusive range [0, p - 1]. In this case, let x_string = I2OSP(x, 48).

*If P is the point at infinity on E', let x_string = I2OSP(0, 96).

*If P is a point on E' other than the point at infinity, then x can be represented as (x_0, x_1) where x_0 and x_1 are elements of GF(p), i.e., integers in the inclusive range [0, p - 1] (see discussion of vector representations above). In this case, let x_string = I2OSP(x_1, 48) || I2OSP(x_0, 48).

Notice that in all of the above cases, the 3 most significant bits of x_string[0] are guaranteed to be 0.

4. If point compression is used, let y_string be the empty string. Otherwise (i.e., when point compression is not used), let y_string be the serialization of y, which is defined in Step 3.

5. Let s_string = x_string || y_string.

6. Set s_string[0] = x_string[0] OR m_byte, where OR is computed bitwise. After this operation, the most significant bit of s_string[0] equals C_bit, the next bit equals I_bit, and the next equals S_bit. (This is true because the three most significant bits of x_string[0] are guaranteed to be zero, as discussed above.)

7. Output s_string.

C.2. Point deserialization procedure

The deserialization procedure is defined as follows for a string s_string. This procedure uses the OS2IP function defined in [RFC8017].

1. Let m_byte = s_string[0] AND 0xE0, where AND is computed bitwise. In other words, the three most significant bits of m_byte equal the three most significant bits of s_string[0], and the remaining bits are 0.

   If m_byte equals any of 0x20, 0x60, or 0xE0, output INVALID and stop decoding.

   Otherwise:

   *Let C_bit equal the most significant bit of m_byte, *Let I_bit equal the second most significant bit of m_byte, and

*Let S_bit equal the third most significant bit of m_byte.

2. If C_bit is 1:

   *If s_string has length 48 bytes, the output point is on the curve E.

   *If s_string has length 96 bytes, the output point is on the curve E'.

   *If s_string has any other length, output INVALID and stop decoding.

   If C_bit is 0:

   *If s_string has length 96 bytes, the output point is on E.

   *If s_string has length 192 bytes, the output point is on E'.

   *If s_string has any other length, output INVALID and stop decoding.

3. Let s_string[0] = s_string[0] AND 0x1F, where AND is computed bitwise. In other words, set the three most significant bits of s_string[0] to 0.

4. If I_bit is 1:

   *If s_string is not the all zeros string, output INVALID and stop decoding.

   *Otherwise (i.e., if s_string is the all zeros string), output the point at infinity on the curve that was determined in step 2 and stop decoding.

   Otherwise, I_bit must be 0. Continue decoding.

5. If C_bit is 0:

   *Let x_string be the first half of s_string.

   *Let y_string be the last half of s_string.

   *Let x = OS2IP(x_string).

   *Let y = OS2IP(y_string).

*If the point P = (x, y) is not a valid point on the curve that was determined in step 2, output INVALID and stop decoding.

*Otherwise, output the point P = (x, y) and stop decoding.

Otherwise, C_bit must be 1. Continue decoding.

6. Let x = OS2IP(s_string).

7. If the curve that was determined in step 2 is E:

   *Let y2 = x^3 + 4 in GF(p).

   *If y2 is not square in GF(p), output INVALID and stop decoding.

   *Otherwise, let y = sqrt(y2) in GF(p) and let Y_bit = sign_GF_p(y).

   Otherwise, (i.e., when the curve that was determined in step 2 is E'):

   *Let y2 = x^3 + 4 * (u + 1) in GF(p^2).

   *If y2 is not square in GF(p^2), output INVALID and stop decoding.

   *Otherwise, let y = sqrt(y2) in GF(p^2) and let Y_bit = sign_GF_p^2(y).

8. If S_bit equals Y_bit, output P = (x, y) and stop decoding. Otherwise, output P = (x, -y) and stop decoding.

Appendix D. Adoption Status of Pairing-Friendly Curves with the 100-bit Security Level

BN curves including BN254 that were estimated as the 128-bit security level before exTNFS ensure no more than the 100-bit security level by the effect of exTNFS. Table 2 summarizes the adoption status of the parameters with a security level lower than the "Arnd 128-bit" range. Please refer the Section 4 for the naming conventions for each curve listed in Table 2.

Category	Name	Supported 100-bit Curves
Standard	ISO/IEC	BN256I
	TCG	BN256I
	FIDO/W3C	BN256I
		BN256D

Category	Name	Supported 100-bit Curves
Library	mcl	BN254N
		BN_SNARK1
	TEPLA	BN254B
		BN254N
	RELIC	BN254N
		BN256D
	AMCL	BN254N
		BN254CX
		BN256I
	Intel IPP	BN256I
	MIRACL	BN254N
		BN254CX
		BN256I
	Adjoint	BN_SNARK1
		BN254B
		BN254N
		BN254S1
		BN254S2
Application	Zcash	BN_SNARK1
	DFINITY	BN254N
		BN_SNARK1

Table 2: Adoption Status of Pairing-Friendly Curves with 100-bit Security Level(Legacy)

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