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RaptorQ Forward Error Correction Scheme for Object Delivery

Abstract

This document describes a Fully-Specified Forward Error Correction (FEC) scheme, corresponding to FEC Encoding ID 6, for the RaptorQ FEC code and its application to reliable delivery of data objects.

RaptorQ codes are a new family of codes that provide superior flexibility, support for larger source block sizes, and better coding efficiency than Raptor codes in [RFC 5053](#). RaptorQ is also a fountain code, i.e., as many encoding symbols as needed can be generated on the fly by the encoder from the source symbols of a source block of data. The decoder is able to recover the source block from almost any set of encoding symbols of sufficient cardinality -- in most cases, a set of cardinality equal to the number of source symbols is sufficient; in rare cases, a set of cardinality slightly more than the number of source symbols is required.

The RaptorQ code described here is a systematic code, meaning that all the source symbols are among the encoding symbols that can be generated.

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[1.](#) Introduction

This document specifies an FEC scheme for the RaptorQ forward error correction code for object delivery applications. The concept of an FEC scheme is defined in [RFC 5052](#) [[RFC5052](#)], and this document follows the format prescribed there and uses the terminology of that document. The RaptorQ code described herein is a next generation of the Raptor code described in [RFC 5053](#) [[RFC5053](#)]. The RaptorQ code provides superior reliability, better coding efficiency, and support for larger source block sizes than the Raptor code of [RFC 5053](#) [[RFC5053](#)]. These improvements simplify the usage of the RaptorQ code in an object delivery Content Delivery Protocol compared to [RFC 5053](#) [[RFC5053](#)]. A detailed mathematical design and analysis of the RaptorQ code together with extensive simulation results are provided in [[RaptorCodes](#)].

The RaptorQ FEC scheme is a Fully-Specified FEC scheme corresponding to FEC Encoding ID 6.

RaptorQ is a fountain code, i.e., as many encoding symbols as needed can be generated on the fly by the encoder from the source symbols of a block. The decoder is able to recover the source block from almost any set of encoding symbols of cardinality only slightly larger than the number of source symbols.

The code described in this document is a systematic code; that is, the original unmodified source symbols, as well as a number of repair symbols, can be sent from sender to receiver. For more background on the use of Forward Error Correction codes in reliable multicast, see [[RFC3453](#)].

[2.](#) Requirements Notation

The key words "MUST", "MUST NOT", "REQUIRED", "SHALL", "SHALL NOT", "SHOULD", "SHOULD NOT", "RECOMMENDED", "MAY", and "OPTIONAL" in this document are to be interpreted as described in [[RFC2119](#)].

The encoded Common FEC Object Transmission Information (OTI) format is shown in Figure 2.

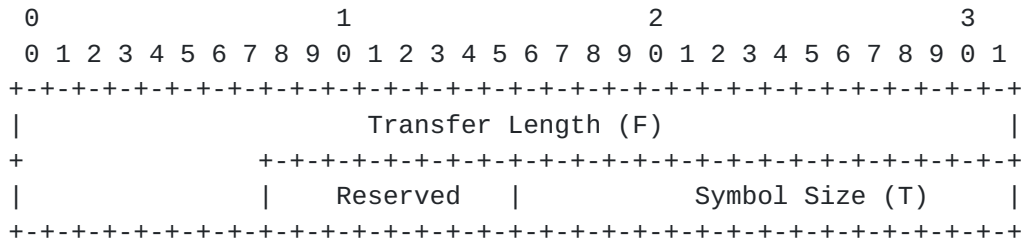


Figure 2: Encoded Common FEC OTI for RaptorQ FEC Scheme

NOTE: The limit of 946270874880 on the transfer length is a consequence of the limitation on the symbol size to $2^{16}-1$, the limitation on the number of symbols in a source block to 56403, and the limitation on the number of source blocks to 2^8 .

3.3.3. Scheme-Specific

The following parameters are carried in the Scheme-Specific FEC Object Transmission Information element for this FEC scheme:

- o The number of source blocks (Z): 8-bit unsigned integer.
- o The number of sub-blocks (N): 16-bit unsigned integer.
- o A symbol alignment parameter (Al): 8-bit unsigned integer.

These parameters are all positive integers. The encoded Scheme-specific Object Transmission Information is a 4-octet field consisting of the parameters Z, N, and Al as shown in Figure 3.

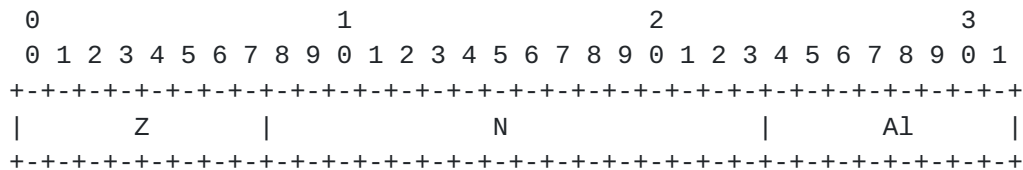


Figure 3: Encoded Scheme-Specific FEC Object Transmission Information

The encoded FEC Object Transmission Information is a 12-octet field consisting of the concatenation of the encoded Common FEC Object Transmission Information and the encoded Scheme-specific FEC Object Transmission Information.

These three parameters define the source block partitioning as described in [Section 4.4.1.2](#).

4. Procedures

4.1. Introduction

For any undefined symbols or functions used in this section, in particular the functions "ceil" and "floor", refer to [Section 5.1](#).

4.2. Content Delivery Protocol Requirements

This section describes the information exchange between the RaptorQ FEC scheme and any Content Delivery Protocol (CDP) that makes use of the RaptorQ FEC scheme for object delivery.

The RaptorQ encoder scheme and RaptorQ decoder scheme for object delivery require the following information from the CDP:

- o F: the transfer length of the object, in octets
- o A1: the symbol alignment parameter
- o T: the symbol size in octets, which MUST be a multiple of A1
- o Z: the number of source blocks
- o N: the number of sub-blocks in each source block

The RaptorQ encoder scheme for object delivery additionally requires:

- the object to be encoded, which is F octets long

The RaptorQ encoder scheme supplies the CDP with the following information for each packet to be sent:

- o Source Block Number (SBN)
- o Encoding Symbol ID (ESI)
- o Encoding symbol(s)

The CDP MUST communicate this information to the receiver.

4.3. Example Parameter Derivation Algorithm

This section provides recommendations for the derivation of the three transport parameters, T, Z, and N. This recommendation is based on the following input parameters:

- o F: the transfer length of the object, in octets

- o WS: the maximum size block that is decodable in working memory, in octets
- o P': the maximum payload size in octets, which is assumed to be a multiple of Al
- o Al: the symbol alignment parameter, in octets
- o SS: a parameter where the desired lower bound on the sub-symbol size is SS*Al
- o K'_max: the maximum number of source symbols per source block.

Note: [Section 5.1.2](#) defines K'_max to be 56403.

Based on the above inputs, the transport parameters T, Z, and N are calculated as follows:

Let

- o $T = P'$
- o $K_t = \text{ceil}(F/T)$
- o $N_{\text{max}} = \text{floor}(T/(SS*Al))$
- o for all $n=1, \dots, N_{\text{max}}$
 - * $KL(n)$ is the maximum K' value in Table 2 in [Section 5.6](#) such that

$$K' \leq WS/(Al*(\text{ceil}(T/(Al*n))))$$

- o $Z = \text{ceil}(K_t/KL(N_{\text{max}}))$
- o N is the minimum $n=1, \dots, N_{\text{max}}$ such that $\text{ceil}(K_t/Z) \leq KL(n)$

It is RECOMMENDED that each packet contains exactly one symbol. However, receivers SHALL support the reception of packets that contain multiple symbols.

The value K_t is the total number of symbols required to represent the source data of the object.

The algorithm above and that defined in [Section 4.4.1.2](#) ensure that the sub-symbol sizes are a multiple of the symbol alignment parameter, Al. This is useful because the sum operations used for encoding and decoding are generally performed several octets at a

time, for example, at least 4 octets at a time on a 32-bit processor. Thus, the encoding and decoding can be performed faster if the sub-symbol sizes are a multiple of this number of octets.

The recommended setting for the input parameter *Al* is 4.

The parameter *WS* can be used to generate encoded data that can be decoded efficiently with limited working memory at the decoder. Note that the actual maximum decoder memory requirement for a given value of *WS* depends on the implementation, but it is possible to implement decoding using working memory only slightly larger than *WS*.

4.4. Object Delivery

4.4.1. Source Block Construction

4.4.1.1. General

In order to apply the RaptorQ encoder to a source object, the object may be broken into $Z \geq 1$ blocks, known as source blocks. The RaptorQ encoder is applied independently to each source block. Each source block is identified by a unique Source Block Number (SBN), where the first source block has SBN zero, the second has SBN one, etc. Each source block is divided into a number, *K*, of source symbols of size *T* octets each. Each source symbol is identified by a unique Encoding Symbol Identifier (ESI), where the first source symbol of a source block has ESI zero, the second has ESI one, etc.

Each source block with *K* source symbols is divided into $N \geq 1$ sub-blocks, which are small enough to be decoded in the working memory. Each sub-block is divided into *K* sub-symbols of size *T'*.

Note that the value of *K* is not necessarily the same for each source block of an object, and the value of *T'* may not necessarily be the same for each sub-block of a source block. However, the symbol size *T* is the same for all source blocks of an object, and the number of symbols *K* is the same for every sub-block of a source block. Exact partitioning of the object into source blocks and sub-blocks is described in [Section 4.4.1.2](#) below.

4.4.1.2. Source Block and Sub-Block Partitioning

The construction of source blocks and sub-blocks is determined based on five input parameters -- *F*, *Al*, *T*, *Z*, and *N* -- and a function `Partition[]`. The five input parameters are defined as follows:

- o *F*: the transfer length of the object, in octets

- o A_l : a symbol alignment parameter, in octets
- o T : the symbol size, in octets, which MUST be a multiple of A_l
- o Z : the number of source blocks
- o N : the number of sub-blocks in each source block

These parameters MUST be set so that $\text{ceil}(\text{ceil}(F/T)/Z) \leq K'_{\text{max}}$. Recommendations for derivation of these parameters are provided in [Section 4.3](#).

The function $\text{Partition}[I, J]$ derives parameters for partitioning a block of size I into J approximately equal-sized blocks. More specifically, it partitions I into J_L blocks of length I_L and J_S blocks of length I_S . The output of $\text{Partition}[I, J]$ is the sequence (I_L, I_S, J_L, J_S) , where $I_L = \text{ceil}(I/J)$, $I_S = \text{floor}(I/J)$, $J_L = I - I_S * J$, and $J_S = J - J_L$.

The source object MUST be partitioned into source blocks and sub-blocks as follows:

Let

- o $K_t = \text{ceil}(F/T)$,
- o $(K_L, K_S, Z_L, Z_S) = \text{Partition}[K_t, Z]$,
- o $(T_L, T_S, N_L, N_S) = \text{Partition}[T/A_l, N]$.

Then, the object MUST be partitioned into $Z = Z_L + Z_S$ contiguous source blocks, the first Z_L source blocks each having $K_L * T$ octets, i.e., K_L source symbols of T octets each, and the remaining Z_S source blocks each having $K_S * T$ octets, i.e., K_S source symbols of T octets each.

If $K_t * T > F$, then, for encoding purposes, the last symbol of the last source block MUST be padded at the end with $K_t * T - F$ zero octets.

Next, each source block with K source symbols MUST be divided into $N = N_L + N_S$ contiguous sub-blocks, the first N_L sub-blocks each consisting of K contiguous sub-symbols of size of $T_L * A_l$ octets and the remaining N_S sub-blocks each consisting of K contiguous sub-symbols of size of $T_S * A_l$ octets. The symbol alignment parameter A_l ensures that sub-symbols are always a multiple of A_l octets.

Finally, the m th symbol of a source block consists of the concatenation of the m th sub-symbol from each of the N sub-blocks. Note that this implies that when $N > 1$, a symbol is NOT a contiguous portion of the object.

4.4.2. Encoding Packet Construction

Each encoding packet contains the following information:

- o Source Block Number (SBN)
- o Encoding Symbol ID (ESI)
- o encoding symbol(s)

Each source block is encoded independently of the others. Each encoding packet contains encoding symbols generated from the one source block identified by the SBN carried in the encoding packet. Source blocks are numbered consecutively from zero.

Encoding Symbol ID values from 0 to $K-1$ identify the source symbols of a source block in sequential order, where K is the number of source symbols in the source block. Encoding Symbol IDs K onwards identify repair symbols generated from the source symbols using the RaptorQ encoder.

Each encoding packet either contains only source symbols (source packet) or contains only repair symbols (repair packet). A packet may contain any number of symbols from the same source block. In the case that the last source symbol in a source packet includes padding octets added for FEC encoding purposes, then these octets need not be included in the packet. Otherwise, each packet MUST contain only whole symbols.

The Encoding Symbol ID, X , carried in each source packet is the Encoding Symbol ID of the first source symbol carried in that packet. The subsequent source symbols in the packet have Encoding Symbol IDs $X+1$ to $X+G-1$ in sequential order, where G is the number of symbols in the packet.

Similarly, the Encoding Symbol ID, X , placed into a repair packet is the Encoding Symbol ID of the first repair symbol in the repair packet, and the subsequent repair symbols in the packet have Encoding Symbol IDs $X+1$ to $X+G-1$ in sequential order, where G is the number of symbols in the packet.

Note that it is not necessary for the receiver to know the total number of repair packets.

4.4.3. Example Receiver Recovery Strategies

A receiver can use the received encoding symbols for each source block of an object to recover the source symbols for that source block independently of all other source blocks.

If there is one sub-block per source block, i.e., $N = 1$, then the portion of the data in the original object in its original order associated with a source block consists of the concatenation of the source symbols of a source block in consecutive ESI order.

If there are multiple sub-blocks per source block, i.e., if $N > 1$, then the portion of the data in the original object in its original order associated with a source block consists of the concatenation of the sub-blocks associated with the source block, where sub-symbols within each sub-block are in consecutive ESI order. In this case, there are different receiver source block recovery strategies worth considering depending on the available amount of Random Access Memory (RAM) at the receiver, as outlined below.

One strategy is to recover the source symbols of a source block using the decoding procedures applied to the received symbols for the source block to recover the source symbols as described in [Section 5](#), and then to reorder the sub-symbols of the source symbols so that all consecutive sub-symbols of the first sub-block are first, followed by all consecutive sub-symbols of the second sub-block, etc., followed by all consecutive sub-symbols of the Nth sub-block. This strategy is especially applicable if the receiver has enough RAM to decode an entire source block.

Another strategy is to separately recover the sub-blocks of a source block. For example, a receiver may demultiplex and store sub-symbols associated with each sub-block separately as packets containing encoding symbols arrive, and then use the stored sub-symbols received for a sub-block to recover that sub-block using the decoding procedures described in [Section 5](#). This strategy is especially applicable if the receiver has enough RAM to decode only one sub-block at a time.

5. RaptorQ FEC Code Specification

5.1. Background

For the purpose of the RaptorQ FEC code specification in this section, the following definitions, symbols, and abbreviations apply. A basic understanding of linear algebra, matrix operations, and finite fields is assumed in this section. In particular, matrix multiplication and matrix inversion operations over a mixture of the

finite fields GF[2] and GF[256] are used. A basic familiarity with sparse linear equations, and efficient implementations of algorithms that take advantage of sparse linear equations, is also quite beneficial to an implementer of this specification.

5.1.1. Definitions

- o Source block: a block of K source symbols that are considered together for RaptorQ encoding and decoding purposes.
- o Extended Source Block: a block of K' source symbols, where $K' \geq K$, constructed from a source block and zero or more padding symbols.
- o Symbol: a unit of data. The size, in octets, of a symbol is known as the symbol size. The symbol size is always a positive integer.
- o Source symbol: the smallest unit of data used during the encoding process. All source symbols within a source block have the same size.
- o Padding symbol: a symbol with all zero bits that is added to the source block to form the extended source block.
- o Encoding symbol: a symbol that can be sent as part of the encoding of a source block. The encoding symbols of a source block consist of the source symbols of the source block and the repair symbols generated from the source block. Repair symbols generated from a source block have the same size as the source symbols of that source block.
- o Repair symbol: the encoding symbols of a source block that are not source symbols. The repair symbols are generated based on the source symbols of a source block.
- o Intermediate symbols: symbols generated from the source symbols using an inverse encoding process based on pre-coding relationships. The repair symbols are then generated directly from the intermediate symbols. The encoding symbols do not include the intermediate symbols, i.e., intermediate symbols are not sent as part of the encoding of a source block. The intermediate symbols are partitioned into LT symbols and PI symbols for the purposes of the encoding process.
- o LT symbols: a process similar to that described in [[LTCodes](#)] is used to generate part of the contribution to each generated encoding symbol from the portion of the intermediate symbols designated as LT symbols.

- o PI symbols: a process even simpler than that described in [LTCodes] is used to generate the other part of the contribution to each generated encoding symbol from the portion of the intermediate symbols designated as PI symbols. In the decoding algorithm suggested in Section 5.4, the PI symbols are inactivated at the start, i.e., are placed into the matrix U at the beginning of the first phase of the decoding algorithm. Because the symbols corresponding to the columns of U are sometimes called the "inactivated" symbols, and since the PI symbols are inactivated at the beginning, they are considered "permanently inactivated".
- o HDPC symbols: there is a small subset of the intermediate symbols that are HDPC symbols. Each HDPC symbol has a pre-coding relationship with a large fraction of the other intermediate symbols. HDPC means "High Density Parity Check".
- o LDPC symbols: there is a moderate-sized subset of the intermediate symbols that are LDPC symbols. Each LDPC symbol has a pre-coding relationship with a small fraction of the other intermediate symbols. LDPC means "Low Density Parity Check".
- o Systematic code: a code in which all source symbols are included as part of the encoding symbols of a source block. The RaptorQ code as described herein is a systematic code.
- o Encoding Symbol ID (ESI): information that uniquely identifies each encoding symbol associated with a source block for sending and receiving purposes.
- o Internal Symbol ID (ISI): information that uniquely identifies each symbol associated with an extended source block for encoding and decoding purposes.
- o Arithmetic operations on octets and symbols and matrices: the operations that are used to produce encoding symbols from source symbols and vice versa. See Section 5.7.

5.1.2. Symbols

- i, j, u, v, h, d, a, b, d1, a1, b1, v, m, x, y represent values or variables of one type or another, depending on the context.
- X denotes a non-negative integer value that is either an ISI value or an ESI value, depending on the context.
- ceil(x) denotes the smallest integer that is greater than or equal to x, where x is a real value.

$\text{floor}(x)$ denotes the largest integer that is less than or equal to x , where x is a real value.

$\text{min}(x,y)$ denotes the minimum value of the values x and y , and in general the minimum value of all the argument values.

$\text{max}(x,y)$ denotes the maximum value of the values x and y , and in general the maximum value of all the argument values.

$i \% j$ denotes i modulo j .

$i + j$ denotes the sum of i and j . If i and j are octets or symbols, this designates the arithmetic on octets or symbols, respectively, as defined in [Section 5.7](#). If i and j are integers, then it denotes the usual integer addition.

$i * j$ denotes the product of i and j . If i and j are octets, this designates the arithmetic on octets, as defined in [Section 5.7](#). If i is an octet and j is a symbol, this denotes the multiplication of a symbol by an octet, as also defined in [Section 5.7](#). Finally, if i and j are integers, $i * j$ denotes the usual product of integers.

$a ^{^b}$ denotes the operation a raised to the power b . If a is an octet and b is a non-negative integer, this is understood to mean $a*a*\dots*a$ (b terms), with $'*'$ being the octet product as defined in [Section 5.7](#).

$u ^{\wedge} v$ denotes, for equal-length bit strings u and v , the bitwise exclusive-or of u and v .

$\text{Transpose}[A]$ denotes the transposed matrix of matrix A . In this specification, all matrices have entries that are octets.

$A^{^{-1}}$ denotes the inverse matrix of matrix A . In this specification, all the matrices have octets as entries, so it is understood that the operations of the matrix entries are to be done as stated in [Section 5.7](#) and $A^{^{-1}}$ is the matrix inverse of A with respect to octet arithmetic.

K denotes the number of symbols in a single source block.

K' denotes the number of source plus padding symbols in an extended source block. For the majority of this specification, the padding symbols are considered to be additional source symbols.

K'_{max} denotes the maximum number of source symbols that can be in a single source block. Set to 56403.

- L denotes the number of intermediate symbols for a single extended source block.
- S denotes the number of LDPC symbols for a single extended source block. These are LT symbols. For each value of K' shown in Table 2 in [Section 5.6](#), the corresponding value of S is a prime number.
- H denotes the number of HDPC symbols for a single extended source block. These are PI symbols.
- B denotes the number of intermediate symbols that are LT symbols excluding the LDPC symbols.
- W denotes the number of intermediate symbols that are LT symbols. For each value of K' in Table 2 shown in [Section 5.6](#), the corresponding value of W is a prime number.
- P denotes the number of intermediate symbols that are PI symbols. These contain all HDPC symbols.
- P_1 denotes the smallest prime number greater than or equal to P .
- U denotes the number of non-HDPC intermediate symbols that are PI symbols.
- C denotes an array of intermediate symbols, $C[0]$, $C[1]$, $C[2]$, ..., $C[L-1]$.
- C' denotes an array of the symbols of the extended source block, where $C'[0]$, $C'[1]$, $C'[2]$, ..., $C'[K-1]$ are the source symbols of the source block and $C'[K]$, $C'[K+1]$, ..., $C'[K'-1]$ are padding symbols.
- V_0 , V_1 , V_2 , V_3 denote four arrays of 32-bit unsigned integers, $V_0[0]$, $V_0[1]$, ..., $V_0[255]$; $V_1[0]$, $V_1[1]$, ..., $V_1[255]$; $V_2[0]$, $V_2[1]$, ..., $V_2[255]$; and $V_3[0]$, $V_3[1]$, ..., $V_3[255]$ as shown in [Section 5.5](#).
- $\text{Rand}[y, i, m]$ denotes a pseudo-random number generator.
- $\text{Deg}[v]$ denotes a degree generator.
- $\text{Enc}[K', C, (d, a, b, d_1, a_1, b_1)]$ denotes an encoding symbol generator.
- $\text{Tuple}[K', X]$ denotes a tuple generator function.

T denotes the symbol size in octets.

$J(K')$ denotes the systematic index associated with K' .

G denotes any generator matrix.

I_S denotes the $S \times S$ identity matrix.

5.2. Overview

This section defines the systematic RaptorQ FEC code.

Symbols are the fundamental data units of the encoding and decoding process. For each source block, all symbols are the same size, referred to as the symbol size T . The atomic operations performed on symbols for both encoding and decoding are the arithmetic operations defined in [Section 5.7](#).

The basic encoder is described in [Section 5.3](#). The encoder first derives a block of intermediate symbols from the source symbols of a source block. This intermediate block has the property that both source and repair symbols can be generated from it using the same process. The encoder produces repair symbols from the intermediate block using an efficient process, where each such repair symbol is the exclusive-or of a small number of intermediate symbols from the block. Source symbols can also be reproduced from the intermediate block using the same process. The encoding symbols are the combination of the source and repair symbols.

An example of a decoder is described in [Section 5.4](#). The process for producing source and repair symbols from the intermediate block is designed so that the intermediate block can be recovered from any sufficiently large set of encoding symbols, independent of the mix of source and repair symbols in the set. Once the intermediate block is recovered, missing source symbols of the source block can be recovered using the encoding process.

Requirements for a RaptorQ-compliant decoder are provided in [Section 5.8](#). A number of decoding algorithms are possible to achieve these requirements. An efficient decoding algorithm to achieve these requirements is provided in [Section 5.4](#).

The construction of the intermediate and repair symbols is based in part on a pseudo-random number generator described in [Section 5.3](#). This generator is based on a fixed set of 1024 random numbers that must be available to both sender and receiver. These numbers are

provided in [Section 5.5](#). Encoding and decoding operations for RaptorQ use operations on octets. [Section 5.7](#) describes how to perform these operations.

Finally, the construction of the intermediate symbols from the source symbols is governed by "systematic indices", values of which are provided in [Section 5.6](#) for specific extended source block sizes between 6 and $K'_{\text{max}} = 56403$ source symbols. Thus, the RaptorQ code supports source blocks with between 1 and 56403 source symbols.

5.3. Systematic RaptorQ Encoder

5.3.1. Introduction

For a given source block of K source symbols, for encoding and decoding purposes, the source block is augmented with $K' - K$ additional padding symbols, where K' is the smallest value that is at least K in the systematic index Table 2 of [Section 5.6](#). The reason for padding out a source block to a multiple of K' is to enable faster encoding and decoding and to minimize the amount of table information that needs to be stored in the encoder and decoder.

For purposes of transmitting and receiving data, the value of K is used to determine the number of source symbols in a source block, and thus K needs to be known at the sender and the receiver. In this case, the sender and receiver can compute K' from K and the $K' - K$ padding symbols can be automatically added to the source block without any additional communication. The encoding symbol ID (ESI) is used by a sender and receiver to identify the encoding symbols of a source block, where the encoding symbols of a source block consist of the source symbols and the repair symbols associated with the source block. For a source block with K source symbols, the ESIs for the source symbols are 0, 1, 2, ..., $K-1$, and the ESIs for the repair symbols are K , $K+1$, $K+2$, Using the ESI for identifying encoding symbols in transport ensures that the ESI values continue consecutively between the source and repair symbols.

For purposes of encoding and decoding data, the value of K' derived from K is used as the number of source symbols of the extended source block upon which encoding and decoding operations are performed, where the K' source symbols consist of the original K source symbols and an additional $K' - K$ padding symbols. The Internal Symbol ID (ISI) is used by the encoder and decoder to identify the symbols associated with the extended source block, i.e., for generating encoding symbols and for decoding. For a source block with K original source symbols, the ISIs for the original source symbols are 0, 1, 2, ..., $K-1$, the ISIs for the $K' - K$ padding symbols are K , $K+1$, $K+2$, ..., $K'-1$, and the ISIs for the repair symbols are K' , $K'+1$, $K'+2$, Using the ISI

for encoding and decoding allows the padding symbols of the extended source block to be treated the same way as other source symbols of the extended source block. Also, it ensures that a given prefix of repair symbols are generated in a consistent way for a given number K' of source symbols in the extended source block, independent of K .

The relationship between the ESIs and the ISIs is simple: the ESIs and the ISIs for the original K source symbols are the same, the $K'-K$ padding symbols have an ISI but do not have a corresponding ESI (since they are symbols that are neither sent nor received), and a repair symbol ISI is simply the repair symbol ESI plus $K'-K$. The translation between ESIs (used to identify encoding symbols sent and received) and the corresponding ISIs (used for encoding and decoding), as well as determining the proper padding of the extended source block with padding symbols (used for encoding and decoding), is the internal responsibility of the RaptorQ encoder/decoder.

5.3.2. Encoding Overview

The systematic RaptorQ encoder is used to generate any number of repair symbols from a source block that consists of K source symbols placed into an extended source block C' . Figure 4 shows the encoding overview.

The first step of encoding is to construct an extended source block by adding zero or more padding symbols such that the total number of symbols, K' , is one of the values listed in [Section 5.6](#). Each padding symbol consists of T octets where the value of each octet is zero. K' MUST be selected as the smallest value of K' from the table of [Section 5.6](#) that is greater than or equal to K .

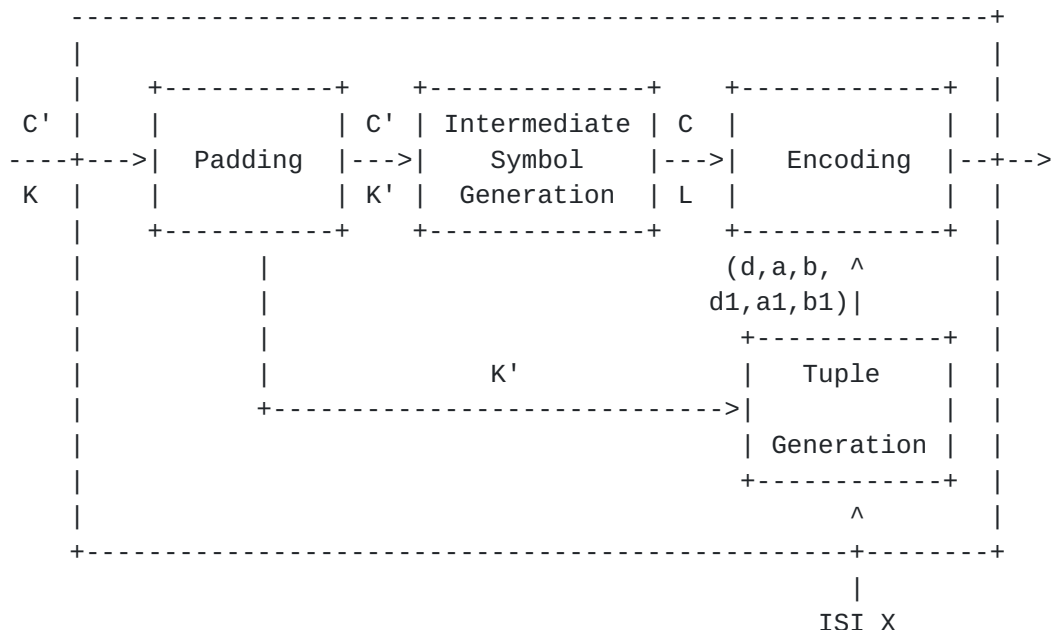


Figure 4: Encoding Overview

Let $C'[0], \dots, C'[K-1]$ denote the K source symbols.

Let $C'[K], \dots, C'[K'-1]$ denote the $K'-K$ padding symbols, which are all set to zero bits. Then, $C'[0], \dots, C'[K'-1]$ are the symbols of the extended source block upon which encoding and decoding are performed.

In the remainder of this description, these padding symbols will be considered as additional source symbols and referred to as such. However, these padding symbols are not part of the encoding symbols, i.e., they are not sent as part of the encoding. At a receiver, the value of K' can be computed based on K , then the receiver can insert $K' - K$ padding symbols at the end of a source block of K' source symbols and recover the remaining K source symbols of the source block from received encoding symbols.

The second step of encoding is to generate a number, $L > K'$, of intermediate symbols from the K' source symbols. In this step, K' source tuples $(d[0], a[0], b[0], d1[0], a1[0], b1[0]), \dots, (d[K'-1], a[K'-1], b[K'-1], d1[K'-1], a1[K'-1], b1[K'-1])$ are generated using the Tuple[] generator as described in [Section 5.3.5.4](#). The K' source tuples and the ISIs associated with the K' source symbols are used to determine L intermediate symbols $C[0], \dots, C[L-1]$ from the source symbols using an inverse encoding process. This process can be realized by a RaptorQ decoding process.

Certain "pre-coding relationships" must hold within the L intermediate symbols. [Section 5.3.3.3](#) describes these relationships. [Section 5.3.3.4](#) describes how the intermediate symbols are generated from the source symbols.

Once the intermediate symbols have been generated, repair symbols can be produced. For a repair symbol with ISI $X > K'$, the tuple of non-negative integers $(d, a, b, d1, a1, b1)$ can be generated, using the `Tuple[]` generator as described in [Section 5.3.5.4](#). Then, the $(d, a, b, d1, a1, b1)$ tuple and the ISI X are used to generate the corresponding repair symbol from the intermediate symbols using the `Enc[]` generator described in [Section 5.3.5.3](#). The corresponding ESI for this repair symbol is then $X - (K' - K)$. Note that source symbols of the extended source block can also be generated using the same process, i.e., for any $X < K'$, the symbol generated using this process has the same value as $C'[X]$.

5.3.3. First Encoding Step: Intermediate Symbol Generation

5.3.3.1. General

This encoding step is a pre-coding step to generate the L intermediate symbols $C[0], \dots, C[L-1]$ from the source symbols $C'[0], \dots, C'[K'-1]$, where $L > K'$ is defined in [Section 5.3.3.3](#). The intermediate symbols are uniquely defined by two sets of constraints:

1. The intermediate symbols are related to the source symbols by a set of source symbol tuples and by the ISIs of the source symbols. The generation of the source symbol tuples is defined in [Section 5.3.3.2](#) using the `Tuple[]` generator as described in [Section 5.3.5.4](#).
2. A number of pre-coding relationships hold within the intermediate symbols themselves. These are defined in [Section 5.3.3.3](#).

The generation of the L intermediate symbols is then defined in [Section 5.3.3.4](#).

5.3.3.2. Source Symbol Tuples

Each of the K' source symbols is associated with a source symbol tuple $(d[X], a[X], b[X], d1[X], a1[X], b1[X])$ for $0 \leq X < K'$. The source symbol tuples are determined using the `Tuple[]` generator defined in [Section 5.3.5.4](#) as:

For each X , $0 \leq X < K'$

$$(d[X], a[X], b[X], d1[X], a1[X], b1[X]) = \text{Tuple}[K, X]$$

5.3.3.3. Pre-Coding Relationships

The pre-coding relationships amongst the L intermediate symbols are defined by requiring that a set of $S+H$ linear combinations of the intermediate symbols evaluate to zero. There are S LDPC and H HDPC symbols, and thus $L = K' + S + H$. Another partition of the L intermediate symbols is into two sets, one set of W LT symbols and another set of P PI symbols, and thus it is also the case that $L = W + P$. The P PI symbols are treated differently than the W LT symbols in the encoding process. The P PI symbols consist of the H HDPC symbols together with a set of $U = P - H$ of the other K' intermediate symbols. The W LT symbols consist of the S LDPC symbols together with $W - S$ of the other K' intermediate symbols. The values of these parameters are determined from K' as described below, where $H(K')$, $S(K')$, and $W(K')$ are derived from Table 2 in [Section 5.6](#).

Let

- o $S = S(K')$
- o $H = H(K')$
- o $W = W(K')$
- o $L = K' + S + H$
- o $P = L - W$
- o P_1 denote the smallest prime number greater than or equal to P .
- o $U = P - H$
- o $B = W - S$
- o $C[0], \dots, C[B-1]$ denote the intermediate symbols that are LT symbols but not LDPC symbols.
- o $C[B], \dots, C[B+S-1]$ denote the S LDPC symbols that are also LT symbols.
- o $C[W], \dots, C[W+U-1]$ denote the intermediate symbols that are PI symbols but not HDPC symbols.
- o $C[L-H], \dots, C[L-1]$ denote the H HDPC symbols that are also PI symbols.

The first set of pre-coding relations, called LDPC relations, is described below and requires that at the end of this process the set of symbols $D[0]$, ..., $D[S-1]$ are all zero:

- o Initialize the symbols $D[0] = C[B]$, ..., $D[S-1] = C[B+S-1]$.
- o For $i = 0$, ..., $B-1$ do
 - * $a = 1 + \text{floor}(i/S)$
 - * $b = i \% S$
 - * $D[b] = D[b] + C[i]$
 - * $b = (b + a) \% S$
 - * $D[b] = D[b] + C[i]$
 - * $b = (b + a) \% S$
 - * $D[b] = D[b] + C[i]$
- o For $i = 0$, ..., $S-1$ do
 - * $a = i \% P$
 - * $b = (i+1) \% P$
 - * $D[i] = D[i] + C[W+a] + C[W+b]$

Recall that the addition of symbols is to be carried out as specified in [Section 5.7](#).

Note that the LDPC relations as defined in the algorithm above are linear, so there exists an $S \times B$ matrix $G_{LDPC,1}$ and an $S \times P$ matrix $G_{LDPC,2}$ such that

$$G_{LDPC,1} * \text{Transpose}[(C[0], \dots, C[B-1])] + G_{LDPC,2} * \text{Transpose}(C[W], \dots, C[W+P-1]) + \text{Transpose}[(C[B], \dots, C[B+S-1])] = 0$$

(The matrix $G_{LDPC,1}$ is defined by the first loop in the above algorithm, and $G_{LDPC,2}$ can be deduced from the second loop.)

The second set of relations among the intermediate symbols $C[0]$, ..., $C[L-1]$ are the HDPC relations and they are defined as follows:

Let

- o α denote the octet represented by integer 2 as defined in [Section 5.7](#).
- o MT denote an $H \times (K' + S)$ matrix of octets, where for $j=0, \dots, K'+S-2$, the entry $MT[i,j]$ is the octet represented by the integer 1 if $i = \text{Rand}[j+1,6,H]$ or $i = (\text{Rand}[j+1,6,H] + \text{Rand}[j+1,7,H-1] + 1) \% H$, and $MT[i,j]$ is the zero element for all other values of i , and for $j=K'+S-1$, $MT[i,j] = \alpha^{i,j}$ for $i=0, \dots, H-1$.
- o GAMMA denote a $(K'+S) \times (K'+S)$ matrix of octets, where

$$\begin{aligned} \text{GAMMA}[i,j] = \\ & \alpha^{i-j} \text{ for } i \geq j, \\ & 0 \text{ otherwise.} \end{aligned}$$

Then, the relationship between the first $K'+S$ intermediate symbols $C[0], \dots, C[K'+S-1]$ and the H HDPC symbols $C[K'+S], \dots, C[K'+S+H-1]$ is given by:

$$\begin{aligned} \text{Transpose}[C[K'+S], \dots, C[K'+S+H-1]] + MT * \text{GAMMA} * \\ \text{Transpose}[C[0], \dots, C[K'+S-1]] = 0, \end{aligned}$$

where '*' represents standard matrix multiplication utilizing the octet multiplication to define the multiplication between a matrix of octets and a matrix of symbols (in particular, the column vector of symbols), and '+' denotes addition over octet vectors.

[5.3.3.4](#). Intermediate Symbols

[5.3.3.4.1](#). Definition

Given the K' source symbols $C'[0], C'[1], \dots, C'[K'-1]$ the L intermediate symbols $C[0], C[1], \dots, C[L-1]$ are the uniquely defined symbol values that satisfy the following conditions:

1. The K' source symbols $C'[0], C'[1], \dots, C'[K'-1]$ satisfy the K' constraints

$$C'[X] = \text{Enc}[K', (C[0], \dots, C[L-1]), (d[X], a[X], b[X], d1[X], a1[X], b1[X])], \text{ for all } X, 0 \leq X < K',$$

where $(d[X], a[X], b[X], d1[X], a1[X], b1[X]) = \text{Tuple}[K', X]$, $\text{Tuple}[]$ is defined in [Section 5.3.5.4](#), and $\text{Enc}[]$ is described in [Section 5.3.5.3](#).

2. The L intermediate symbols $C[0], C[1], \dots, C[L-1]$ satisfy the pre-coding relationships defined in [Section 5.3.3.3](#).

5.3.3.4.2. Example Method for Calculation of Intermediate Symbols

This section describes a possible method for calculation of the L intermediate symbols $C[0], C[1], \dots, C[L-1]$ satisfying the constraints in [Section 5.3.3.4.1](#).

The L intermediate symbols can be calculated as follows:

Let

- o C denote the column vector of the L intermediate symbols, $C[0], C[1], \dots, C[L-1]$.
- o D denote the column vector consisting of $S+H$ zero symbols followed by the K' source symbols $C'[0], C'[1], \dots, C'[K'-1]$.

Then, the above constraints define an $L \times L$ matrix A of octets such that:

$$A \cdot C = D$$

The matrix A can be constructed as follows:

Let

- o $G_{LDPC,1}$ and $G_{LDPC,2}$ be $S \times B$ and $S \times P$ matrices as defined in [Section 5.3.3.3](#).
- o G_{HDPC} be the $H \times (K'+S)$ matrix such that

$$G_{HDPC} * \text{Transpose}(C[0], \dots, C[K'+S-1]) = \text{Transpose}(C[K'+S], \dots, C[L-1]),$$
 i.e., $G_{HDPC} = M^T * \text{GAMMA}$
- o I_S be the $S \times S$ identity matrix
- o I_H be the $H \times H$ identity matrix
- o G_{ENC} be the $K' \times L$ matrix such that

$$G_{ENC} * \text{Transpose}[(C[0], \dots, C[L-1])] = \text{Transpose}[(C'[0], C'[1], \dots, C'[K'-1])],$$

i.e., $G_ENC[i,j] = 1$ if and only if $C[j]$ is included in the symbols that are summed to produce $Enc[K']$, $(C[0], \dots, C[L-1])$, $(d[i], a[i], b[i], d1[i], a1[i], b1[i])$ and $G_ENC[i,j] = 0$ otherwise.

Then

- o The first S rows of A are equal to $G_LDPC,1 \mid I_S \mid G_LDPC,2$.
- o The next H rows of A are equal to $G_HDPC \mid I_H$.
- o The remaining K' rows of A are equal to G_ENC .

The matrix A is depicted in Figure 5 below:

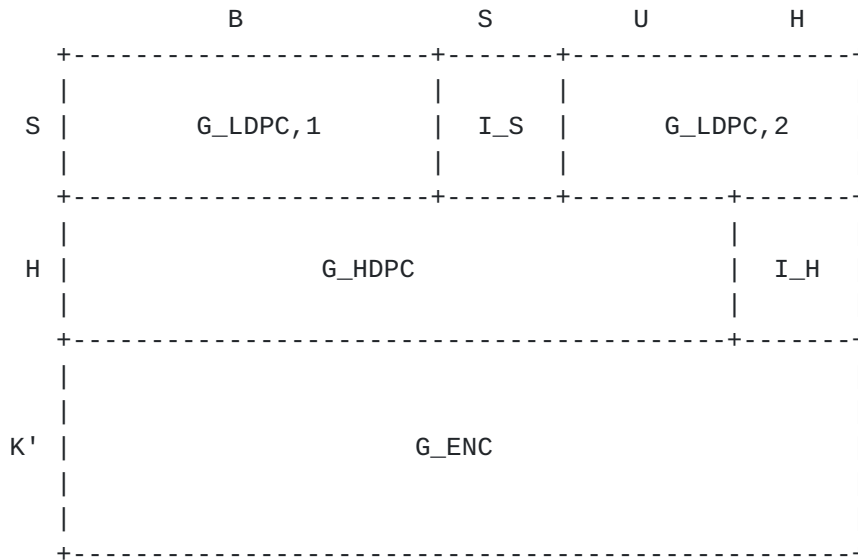


Figure 5: The Matrix A

The intermediate symbols can then be calculated as:

$$C = (A^{-1}) * D$$

The source tuples are generated such that for any K' matrix A has full rank and is therefore invertible. This calculation can be realized by applying a RaptorQ decoding process to the K' source symbols $C'[0], C'[1], \dots, C'[K'-1]$ to produce the L intermediate symbols $C[0], C[1], \dots, C[L-1]$.

To efficiently generate the intermediate symbols from the source symbols, it is recommended that an efficient decoder implementation such as that described in [Section 5.4](#) be used.

5.3.4. Second Encoding Step: Encoding

In the second encoding step, the repair symbol with ISI X ($X \geq K'$) is generated by applying the generator $\text{Enc}[K', (C[0], C[1], \dots, C[L-1]), (d, a, b, d1, a1, b1)]$ defined in [Section 5.3.5.3](#) to the L intermediate symbols $C[0], C[1], \dots, C[L-1]$ using the tuple $(d, a, b, d1, a1, b1) = \text{Tuple}[K', X]$.

5.3.5. Generators

5.3.5.1. Random Number Generator

The random number generator $\text{Rand}[y, i, m]$ is defined as follows, where y is a non-negative integer, i is a non-negative integer less than 256, and m is a positive integer, and the value produced is an integer between 0 and $m-1$. Let $V0, V1, V2$, and $V3$ be the arrays provided in [Section 5.5](#).

Let

- o $x0 = (y + i) \bmod 2^{28}$
- o $x1 = (\text{floor}(y / 2^{28}) + i) \bmod 2^{28}$
- o $x2 = (\text{floor}(y / 2^{16}) + i) \bmod 2^{28}$
- o $x3 = (\text{floor}(y / 2^{24}) + i) \bmod 2^{28}$

Then

$$\text{Rand}[y, i, m] = (V0[x0] \wedge V1[x1] \wedge V2[x2] \wedge V3[x3]) \% m$$

5.3.5.2. Degree Generator

The degree generator $\text{Deg}[v]$ is defined as follows, where v is a non-negative integer that is less than $2^{20} = 1048576$. Given v , find index d in Table 1 such that $f[d-1] \leq v < f[d]$, and set $\text{Deg}[v] = \min(d, W-2)$. Recall that W is derived from K' as described in [Section 5.3.3.3](#).

Index d	f[d]	Index d	f[d]
0	0	1	5243
2	529531	3	704294
4	791675	5	844104
6	879057	7	904023
8	922747	9	937311
10	948962	11	958494
12	966438	13	973160
14	978921	15	983914
16	988283	17	992138
18	995565	19	998631
20	1001391	21	1003887
22	1006157	23	1008229
24	1010129	25	1011876
26	1013490	27	1014983
28	1016370	29	1017662
30	1048576		

Table 1: Defines the Degree Distribution for Encoding Symbols

5.3.5.3. Encoding Symbol Generator

The encoding symbol generator $\text{Enc}[K', (C[0], C[1], \dots, C[L-1]), (d, a, b, d1, a1, b1)]$ takes the following inputs:

- o K' is the number of source symbols for the extended source block. Let L, W, B, S, P , and $P1$ be derived from K' as described in [Section 5.3.3.3](#).

- o $(C[0], C[1], \dots, C[L-1])$ is the array of L intermediate symbols (sub-symbols) generated as described in [Section 5.3.3.4](#).
- o $(d, a, b, d1, a1, b1)$ is a source tuple determined from ISI X using the Tuple[] generator defined in [Section 5.3.5.4](#), whereby
 - * d is a positive integer denoting an encoding symbol LT degree
 - * a is a positive integer between 1 and $W-1$ inclusive
 - * b is a non-negative integer between 0 and $W-1$ inclusive
 - * $d1$ is a positive integer that has value either 2 or 3 denoting an encoding symbol PI degree
 - * $a1$ is a positive integer between 1 and $P1-1$ inclusive
 - * $b1$ is a non-negative integer between 0 and $P1-1$ inclusive

The encoding symbol generator produces a single encoding symbol as output (referred to as result), according to the following algorithm:

- o $result = C[b]$
- o For $j = 1, \dots, d-1$ do
 - * $b = (b + a) \% W$
 - * $result = result + C[b]$
- o While $(b1 \geq P)$ do $b1 = (b1+a1) \% P1$
- o $result = result + C[W+b1]$
- o For $j = 1, \dots, d1-1$ do
 - * $b1 = (b1 + a1) \% P1$
 - * While $(b1 \geq P)$ do $b1 = (b1+a1) \% P1$
 - * $result = result + C[W+b1]$
- o Return result

5.3.5.4. Tuple Generator

The tuple generator $\text{Tuple}[K', X]$ takes the following inputs:

- o K' : the number of source symbols in the extended source block
- o X : an ISI

Let

- o L be determined from K' as described in [Section 5.3.3.3](#)
- o $J = J(K')$ be the systematic index associated with K' , as defined in Table 2 in [Section 5.6](#)

The output of the tuple generator is a tuple, $(d, a, b, d1, a1, b1)$, determined as follows:

- o $A = 53591 + J * 997$
- o if $(A \% 2 == 0)$ { $A = A + 1$ }
- o $B = 10267 * (J + 1)$
- o $y = (B + X * A) \% 2^{32}$
- o $v = \text{Rand}[y, 0, 2^{20}]$
- o $d = \text{Deg}[v]$
- o $a = 1 + \text{Rand}[y, 1, W - 1]$
- o $b = \text{Rand}[y, 2, W]$
- o If $(d < 4)$ { $d1 = 2 + \text{Rand}[X, 3, 2]$ } else { $d1 = 2$ }
- o $a1 = 1 + \text{Rand}[X, 4, P1 - 1]$
- o $b1 = \text{Rand}[X, 5, P1]$

5.4. Example FEC Decoder

5.4.1. General

This section describes an efficient decoding algorithm for the RaptorQ code introduced in this specification. Note that each received encoding symbol is a known linear combination of the intermediate symbols. So, each received encoding symbol provides a

linear equation among the intermediate symbols, which, together with the known linear pre-coding relationships amongst the intermediate symbols, gives a system of linear equations. Thus, any algorithm for solving systems of linear equations can successfully decode the intermediate symbols and hence the source symbols. However, the algorithm chosen has a major effect on the computational efficiency of the decoding.

5.4.2. Decoding an Extended Source Block

5.4.2.1. General

It is assumed that the decoder knows the structure of the source block it is to decode, including the symbol size, T , and the number K of symbols in the source block and the number K' of source symbols in the extended source block.

From the algorithms described in [Section 5.3](#), the RaptorQ decoder can calculate the total number $L = K' + S + H$ of intermediate symbols and determine how they were generated from the extended source block to be decoded. In this description, it is assumed that the received encoding symbols for the extended source block to be decoded are passed to the decoder. Furthermore, for each such encoding symbol, it is assumed that the number and set of intermediate symbols whose sum is equal to the encoding symbol are passed to the decoder. In the case of source symbols, including padding symbols, the source symbol tuples described in [Section 5.3.3.2](#) indicate the number and set of intermediate symbols that sum to give each source symbol.

Let $N \geq K'$ be the number of received encoding symbols to be used for decoding, including padding symbols for an extended source block, and let $M = S + H + N$. Then, with the notation of [Section 5.3.3.4.2](#), we have $A \cdot C = D$.

Decoding an extended source block is equivalent to decoding C from known A and D . It is clear that C can be decoded if and only if the rank of A is L . Once C has been decoded, missing source symbols can be obtained by using the source symbol tuples to determine the number and set of intermediate symbols that must be summed to obtain each missing source symbol.

The first step in decoding C is to form a decoding schedule. In this step, A is converted using Gaussian elimination (using row operations and row and column reorderings) and after discarding $M - L$ rows, into the $L \times L$ identity matrix. The decoding schedule consists of the sequence of row operations and row and column reorderings during the Gaussian elimination process, and it only depends on A and not on D .

The decoding of C from D can take place concurrently with the forming of the decoding schedule, or the decoding can take place afterwards based on the decoding schedule.

The correspondence between the decoding schedule and the decoding of C is as follows. Let $c[0] = 0$, $c[1] = 1$, ..., $c[L-1] = L-1$ and $d[0] = 0$, $d[1] = 1$, ..., $d[M-1] = M-1$ initially.

- o Each time a multiple, β , of row i of A is added to row i' in the decoding schedule, then in the decoding process the symbol $\beta \cdot D[d[i]]$ is added to symbol $D[d[i']]$.
- o Each time a row i of A is multiplied by an octet β , then in the decoding process the symbol $D[d[i]]$ is also multiplied by β .
- o Each time row i is exchanged with row i' in the decoding schedule, then in the decoding process the value of $d[i]$ is exchanged with the value of $d[i']$.
- o Each time column j is exchanged with column j' in the decoding schedule, then in the decoding process the value of $c[j]$ is exchanged with the value of $c[j']$.

From this correspondence, it is clear that the total number of operations on symbols in the decoding of the extended source block is the number of row operations (not exchanges) in the Gaussian elimination. Since A is the $L \times L$ identity matrix after the Gaussian elimination and after discarding the last $M - L$ rows, it is clear at the end of successful decoding that the L symbols $D[d[0]]$, $D[d[1]]$, ..., $D[d[L-1]]$ are the values of the L symbols $C[c[0]]$, $C[c[1]]$, ..., $C[c[L-1]]$.

The order in which Gaussian elimination is performed to form the decoding schedule has no bearing on whether or not the decoding is successful. However, the speed of the decoding depends heavily on the order in which Gaussian elimination is performed. (Furthermore, maintaining a sparse representation of A is crucial, although this is not described here.) The remainder of this section describes an order in which Gaussian elimination could be performed that is relatively efficient.

5.4.2.2. First Phase

In the first phase of the Gaussian elimination, the matrix A is conceptually partitioned into submatrices and, additionally, a matrix X is created. This matrix has as many rows and columns as A , and it will be a lower triangular matrix throughout the first phase. At the beginning of this phase, the matrix A is copied into the matrix X .

The submatrix sizes are parameterized by non-negative integers i and u , which are initialized to 0 and P , the number of PI symbols, respectively. The submatrices of A are:

1. The submatrix I defined by the intersection of the first i rows and first i columns. This is the identity matrix at the end of each step in the phase.
2. The submatrix defined by the intersection of the first i rows and all but the first i columns and last u columns. All entries of this submatrix are zero.
3. The submatrix defined by the intersection of the first i columns and all but the first i rows. All entries of this submatrix are zero.
4. The submatrix U defined by the intersection of all the rows and the last u columns.
5. The submatrix V formed by the intersection of all but the first i columns and the last u columns and all but the first i rows.

Figure 6 illustrates the submatrices of A . At the beginning of the first phase, V consists of the first $L-P$ columns of A , and U consists of the last P columns corresponding to the PI symbols. In each step, a row of A is chosen.

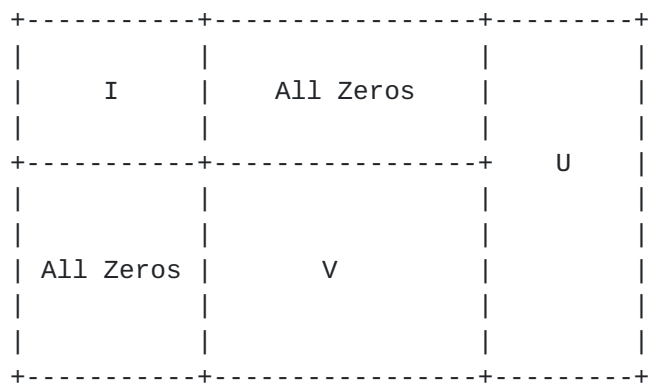


Figure 6: Submatrices of A in the First Phase

The following graph defined by the structure of V is used in determining which row of A is chosen. The columns that intersect V are the nodes in the graph, and the rows that have exactly 2 nonzero entries in V and are not HDPC rows are the edges of the graph that connect the two columns (nodes) in the positions of the two ones. A component in this graph is a maximal set of nodes (columns) and edges

(rows) such that there is a path between each pair of nodes/edges in the graph. The size of a component is the number of nodes (columns) in the component.

There are at most L steps in the first phase. The phase ends successfully when $i + u = L$, i.e., when V and the all zeros submatrix above V have disappeared, and A consists of I , the all zeros submatrix below I , and U . The phase ends unsuccessfully in decoding failure if at some step before V disappears there is no nonzero row in V to choose in that step. In each step, a row of A is chosen as follows:

- o If all entries of V are zero, then no row is chosen and decoding fails.
- o Let r be the minimum integer such that at least one row of A has exactly r nonzeros in V .
 - * If $r \neq 2$, then choose a row with exactly r nonzeros in V with minimum original degree among all such rows, except that HDPC rows should not be chosen until all non-HDPC rows have been processed.
 - * If $r = 2$ and there is a row with exactly 2 ones in V , then choose any row with exactly 2 ones in V that is part of a maximum size component in the graph described above that is defined by V .
 - * If $r = 2$ and there is no row with exactly 2 ones in V , then choose any row with exactly 2 nonzeros in V .

After the row is chosen in this step, the first row of A that intersects V is exchanged with the chosen row so that the chosen row is the first row that intersects V . The columns of A among those that intersect V are reordered so that one of the r nonzeros in the chosen row appears in the first column of V and so that the remaining $r-1$ nonzeros appear in the last columns of V . The same row and column operations are also performed on the matrix X . Then, an appropriate multiple of the chosen row is added to all the other rows of A below the chosen row that have a nonzero entry in the first column of V . Specifically, if a row below the chosen row has entry β in the first column of V , and the chosen row has entry α in the first column of V , then β/α multiplied by the chosen row is added to this row to leave a zero value in the first column of V .

Finally, i is incremented by 1 and u is incremented by $r-1$, which completes the step.

Note that efficiency can be improved if the row operations identified above are not actually performed until the affected row is itself chosen during the decoding process. This avoids processing of row operations for rows that are not eventually used in the decoding process, and in particular this avoids those rows for which $\beta \neq 1$ until they are actually required. Furthermore, the row operations required for the HDPC rows may be performed for all such rows in one process, by using the algorithm described in [Section 5.3.3.3](#).

[5.4.2.3](#). Second Phase

At this point, all the entries of X outside the first i rows and i columns are discarded, so that X has lower triangular form. The last i rows and columns of X are discarded, so that X now has i rows and i columns. The submatrix U is further partitioned into the first i rows, U_{upper} , and the remaining $M - i$ rows, U_{lower} . Gaussian elimination is performed in the second phase on U_{lower} either to determine that its rank is less than u (decoding failure) or to convert it into a matrix where the first u rows is the identity matrix (success of the second phase). Call this $u \times u$ identity matrix I_u . The $M - L$ rows of A that intersect $U_{\text{lower}} - I_u$ are discarded. After this phase, A has L rows and L columns.

[5.4.2.4](#). Third Phase

After the second phase, the only portion of A that needs to be zeroed out to finish converting A into the $L \times L$ identity matrix is U_{upper} . The number of rows i of the submatrix U_{upper} is generally much larger than the number of columns u of U_{upper} . Moreover, at this time, the matrix U_{upper} is typically dense, i.e., the number of nonzero entries of this matrix is large. To reduce this matrix to a sparse form, the sequence of operations performed to obtain the matrix U_{lower} needs to be inverted. To this end, the matrix X is multiplied with the submatrix of A consisting of the first i rows of A . After this operation, the submatrix of A consisting of the intersection of the first i rows and columns equals to X , whereas the matrix U_{upper} is transformed to a sparse form.

[5.4.2.5](#). Fourth Phase

For each of the first i rows of U_{upper} , do the following: if the row has a nonzero entry at position j , and if the value of that nonzero entry is b , then add to this row b times row j of I_u . After this step, the submatrix of A consisting of the intersection of the first i rows and columns is equal to X , the submatrix U_{upper} consists of zeros, the submatrix consisting of the intersection of the last u rows and the first i columns consists of zeros, and the submatrix consisting of the last u rows and columns is the matrix I_u .

5.4.2.6. Fifth Phase

For j from 1 to i , perform the following operations:

1. If $A[j,j]$ is not one, then divide row j of A by $A[j,j]$.
2. For l from 1 to $j-1$, if $A[j,l]$ is nonzero, then add $A[j,l]$ multiplied with row l of A to row j of A .

After this phase, A is the $L \times L$ identity matrix and a complete decoding schedule has been successfully formed. Then, the corresponding decoding consisting of summing known encoding symbols can be executed to recover the intermediate symbols based on the decoding schedule. The tuples associated with all source symbols are computed according to [Section 5.3.3.2](#). The tuples for received source symbols are used in the decoding. The tuples for missing source symbols are used to determine which intermediate symbols need to be summed to recover the missing source symbols.

5.5. Random Numbers

The four arrays $V0$, $V1$, $V2$, and $V3$ used in [Section 5.3.5.1](#) are provided below. There are 256 entries in each of the four arrays. The indexing into each array starts at 0, and the entries are 32-bit unsigned integers.

5.5.1. The Table $V0$

251291136, 3952231631, 3370958628, 4070167936, 123631495,
 3351110283, 3218676425, 2011642291, 774603218, 2402805061,
 1004366930, 1843948209, 428891132, 3746331984, 1591258008,
 3067016507, 1433388735, 504005498, 2032657933, 3419319784,
 2805686246, 3102436986, 3808671154, 2501582075, 3978944421,
 246043949, 4016898363, 649743608, 1974987508, 2651273766,
 2357956801, 689605112, 715807172, 2722736134, 191939188,
 3535520147, 3277019569, 1470435941, 3763101702, 3232409631,
 122701163, 3920852693, 782246947, 372121310, 2995604341,
 2045698575, 2332962102, 4005368743, 218596347, 3415381967,
 4207612806, 861117671, 3676575285, 2581671944, 3312220480,
 681232419, 307306866, 4112503940, 1158111502, 709227802,
 2724140433, 4201101115, 4215970289, 4048876515, 3031661061,
 1909085522, 510985033, 1361682810, 129243379, 3142379587,
 2569842483, 3033268270, 1658118006, 932109358, 1982290045,
 2983082771, 3007670818, 3448104768, 683749698, 778296777,
 1399125101, 1939403708, 1692176003, 3868299200, 1422476658,
 593093658, 1878973865, 2526292949, 1591602827, 3986158854,
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1523670218, 2708475297, 1046771089, 2229796016, 1255426612,
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515623176, 3457502702, 2115821274, 2720124766, 3242576090,
854310108, 425973987, 325832382, 1796851292, 2462744411,
1976681690, 1408671665, 1228817808, 3917210003, 263976645,
2593736473, 2471651269, 4291353919, 650792940, 1191583883,
3046561335, 2466530435, 2545983082, 969168436, 2019348792,
2268075521, 1169345068, 3250240009, 3963499681, 2560755113,
911182396, 760842409, 3569308693, 2687243553, 381854665,
2613828404, 2761078866, 1456668111, 883760091, 3294951678,
1604598575, 1985308198, 1014570543, 2724959607, 3062518035,
3115293053, 138853680, 4160398285, 3322241130, 2068983570,
2247491078, 3669524410, 1575146607, 828029864, 3732001371,
3422026452, 3370954177, 4006626915, 543812220, 1243116171,
3928372514, 2791443445, 4081325272, 2280435605, 885616073,
616452097, 3188863436, 2780382310, 2340014831, 1208439576,
258356309, 3837963200, 2075009450, 3214181212, 3303882142,
880813252, 1355575717, 207231484, 2420803184, 358923368,
1617557768, 3272161958, 1771154147, 2842106362, 1751209208,
1421030790, 658316681, 194065839, 3241510581, 38625260,
301875395, 4176141739, 297312930, 2137802113, 1502984205,
3669376622, 3728477036, 234652930, 2213589897, 2734638932,
1129721478, 3187422815, 2859178611, 3284308411, 3819792700,
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4012642001, 2217188075, 3756943137, 3077882590, 2054995199,
3081443129, 3895398812, 1141097543, 2376261053, 2626898255,
2554703076, 401233789, 1460049922, 678083952, 1064990737,
940909784, 1673396780, 528881783, 1712547446, 3629685652,
1358307511

5.5.2. The Table V1

807385413, 2043073223, 3336749796, 1302105833, 2278607931,
541015020, 1684564270, 372709334, 3508252125, 1768346005,
1270451292, 2603029534, 2049387273, 3891424859, 2152948345,
4114760273, 915180310, 3754787998, 700503826, 2131559305,
1308908630, 224437350, 4065424007, 3638665944, 1679385496,
3431345226, 1779595665, 3068494238, 1424062773, 1033448464,
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259047959, 4057180909, 1575367248, 4151214153, 110249784,
3006865921, 4293710613, 3501256572, 998007483, 499288295,
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2686640734, 2394526209, 2521660077, 49993987, 3843885867,
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1744097284, 579965637, 2037662632, 852173610, 2681403713,

1047144830, 2982173936, 910285038, 4187576520, 2589870048,
989448887, 3292758024, 506322719, 176010738, 1865471968,
2619324712, 564829442, 1996870325, 339697593, 4071072948,
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3237929991, 3852995239, 2398693510, 3754138664, 522074127,
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976628269, 3123492423, 3041418372, 2258059298, 2139377204,
3243642973, 3226247917, 3674004636, 2698992189, 3453843574,
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4161315584, 1273113343, 3269644828, 3681293816, 412536684,
1156034077, 3823026442, 1066971017, 3598330293, 1979273937,
2079029895, 1195045909, 1071986421, 2712821515, 3377754595,
2184151095, 750918864, 2585729879, 4249895712, 1832579367,
1192240192, 946734366, 31230688, 3174399083, 3549375728,
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3122860549, 666423581, 644189126, 226475395, 307789415,
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4259411376, 3323683436, 568512177, 3736601419, 1800276898,
4012458395, 1823982, 27980198, 2023839966, 869505096,
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3171962488, 442160826, 198349622, 45942637, 1324086311,
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3640121682, 3545931032, 2102949142, 2828208598, 3603378023,
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5.5.3. The Table V2

1629829892, 282540176, 2794583710, 496504798, 2990494426,
3070701851, 2575963183, 4094823972, 2775723650, 4079480416,
176028725, 2246241423, 3732217647, 2196843075, 1306949278,
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1891731298, 2368003842, 3537588307, 2969158410, 4230745262,
831906319, 2935838131, 264029468, 120852739, 3200326460,
355445271, 2296305141, 1566296040, 1760127056, 20073893,
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3346087406, 2690756527, 99815156, 4248519977, 2253762642,
3274144518, 598024568, 3299672435, 556579346, 4121041856,
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200201308, 2801945841, 1621715769, 483977159, 423502325,
3689396064, 1850168397, 3359959416, 3459831930, 841488699,
3570506095, 930267420, 1564520841, 2505122797, 593824107,
1116572080, 819179184, 3139123629, 1414339336, 1076360795,
512403845, 177759256, 1701060666, 2239736419, 515179302,
2935012727, 3821357612, 1376520851, 2700745271, 966853647,
1041862223, 715860553, 171592961, 1607044257, 1227236688,
3647136358, 1417559141, 4087067551, 2241705880, 4194136288,
1439041934, 20464430, 119668151, 2021257232, 2551262694,
1381539058, 4082839035, 498179069, 311508499, 3580908637,
2889149671, 142719814, 1232184754, 3356662582, 2973775623,
1469897084, 1728205304, 1415793613, 50111003, 3133413359,
4074115275, 2710540611, 2700083070, 2457757663, 2612845330,
3775943755, 2469309260, 2560142753, 3020996369, 1691667711,
4219602776, 1687672168, 1017921622, 2307642321, 368711460,
3282925988, 213208029, 4150757489, 3443211944, 2846101972,
4106826684, 4272438675, 2199416468, 3710621281, 497564971,
285138276, 765042313, 916220877, 3402623607, 2768784621,
1722849097, 3386397442, 487920061, 3569027007, 3424544196,
217781973, 2356938519, 3252429414, 145109750, 2692588106,
2454747135, 1299493354, 4120241887, 2088917094, 932304329,
1442609203, 952586974, 3509186750, 753369054, 854421006,
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373532666, 2454243090, 2371530493, 3651087521, 2619878153,
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2552507603, 2635912901, 895130484, 3287782244, 3098973502,
990078774, 3780326506, 2290845203, 41729428, 1949580860,
2283959805, 1036946170, 1694887523, 4880696, 466000198,
2765355283, 3318686998, 1266458025, 3919578154, 3545413527,
2627009988, 3744680394, 1696890173, 3250684705, 4142417708,
915739411, 3308488877, 1289361460, 2942552331, 1169105979,
3342228712, 698560958, 1356041230, 2401944293, 107705232,
3701895363, 903928723, 3646581385, 844950914, 1944371367,
3863894844, 2946773319, 1972431613, 1706989237, 29917467,
3497665928

5.5.4. The Table V3

1191369816, 744902811, 2539772235, 3213192037, 3286061266,
1200571165, 2463281260, 754888894, 714651270, 1968220972,
3628497775, 1277626456, 1493398934, 364289757, 2055487592,
3913468088, 2930259465, 902504567, 3967050355, 2056499403,
692132390, 186386657, 832834706, 859795816, 1283120926,
2253183716, 3003475205, 1755803552, 2239315142, 4271056352,
2184848469, 769228092, 1249230754, 1193269205, 2660094102,
642979613, 1687087994, 2726106182, 446402913, 4122186606,
3771347282, 37667136, 192775425, 3578702187, 1952659096,
3989584400, 3069013882, 2900516158, 4045316336, 3057163251,
1702104819, 4116613420, 3575472384, 2674023117, 1409126723,
3215095429, 1430726429, 2544497368, 1029565676, 1855801827,
4262184627, 1854326881, 2906728593, 3277836557, 2787697002,
2787333385, 3105430738, 2477073192, 748038573, 1088396515,
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2289874706, 2252450179, 944880097, 371933491, 1649074501,
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3033612415, 2311650167, 1048129133, 4206710184, 4209176741,
2640950279, 2096382177, 4116899089, 3631017851, 4104488173,
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303042654, 3990027945, 1562756376, 3413341792, 2059647769,

2823844432, 674595301, 2457639984, 4076754716, 2447737904,
 1583323324, 625627134, 3076006391, 345777990, 1684954145,
 879227329, 3436182180, 1522273219, 3802543817, 1456017040,
 1897819847, 2970081129, 1382576028, 3820044861, 1044428167,
 612252599, 3340478395, 2150613904, 3397625662, 3573635640,
 3432275192

5.6. Systematic Indices and Other Parameters

Table 2 below specifies the supported values of K' . The table also specifies for each supported value of K' the systematic index $J(K')$, the number $H(K')$ of HDPC symbols, the number $S(K')$ of LDPC symbols, and the number $W(K')$ of LT symbols. For each value of K' , the corresponding values of $S(K')$ and $W(K')$ are prime numbers.

The systematic index $J(K')$ is designed to have the property that the set of source symbol tuples $(d[0], a[0], b[0], d1[0], a1[0], b1[0])$, ..., $(d[K'-1], a[K'-1], b[K'-1], d1[K'-1], a1[K'-1], b1[K'-1])$ are such that the L intermediate symbols are uniquely defined, i.e., the matrix A in Figure 6 has full rank and is therefore invertible.

K'	$J(K')$	$S(K')$	$H(K')$	$W(K')$
10	254	7	10	17
12	630	7	10	19
18	682	11	10	29
20	293	11	10	31
26	80	11	10	37
30	566	11	10	41
32	860	11	10	43
36	267	11	10	47
42	822	11	10	53
46	506	13	10	59
48	589	13	10	61
49	87	13	10	61

55	520	13	10	67
60	159	13	10	71
62	235	13	10	73
69	157	13	10	79
75	502	17	10	89
84	334	17	10	97
88	583	17	10	101
91	66	17	10	103
95	352	17	10	107
97	365	17	10	109
101	562	17	10	113
114	5	19	10	127
119	603	19	10	131
125	721	19	10	137
127	28	19	10	139
138	660	19	10	149
140	829	19	10	151
149	900	23	10	163
153	930	23	10	167
160	814	23	10	173
166	661	23	10	179
168	693	23	10	181
179	780	23	10	191

181	605	23	10	193	
185	551	23	10	197	
187	777	23	10	199	
200	491	23	10	211	
213	396	23	10	223	
217	764	29	10	233	
225	843	29	10	241	
236	646	29	10	251	
242	557	29	10	257	
248	608	29	10	263	
257	265	29	10	271	
263	505	29	10	277	
269	722	29	10	283	
280	263	29	10	293	
295	999	29	10	307	
301	874	29	10	313	
305	160	29	10	317	
324	575	31	10	337	
337	210	31	10	349	
341	513	31	10	353	
347	503	31	10	359	
355	558	31	10	367	
362	932	31	10	373	

368	404	31	10	379
372	520	37	10	389
380	846	37	10	397
385	485	37	10	401
393	728	37	10	409
405	554	37	10	421
418	471	37	10	433
428	641	37	10	443
434	732	37	10	449
447	193	37	10	461
453	934	37	10	467
466	864	37	10	479
478	790	37	10	491
486	912	37	10	499
491	617	37	10	503
497	587	37	10	509
511	800	37	10	523
526	923	41	10	541
532	998	41	10	547
542	92	41	10	557
549	497	41	10	563
557	559	41	10	571
563	667	41	10	577

573	912	41	10	587
580	262	41	10	593
588	152	41	10	601
594	526	41	10	607
600	268	41	10	613
606	212	41	10	619
619	45	41	10	631
633	898	43	10	647
640	527	43	10	653
648	558	43	10	661
666	460	47	10	683
675	5	47	10	691
685	895	47	10	701
693	996	47	10	709
703	282	47	10	719
718	513	47	10	733
728	865	47	10	743
736	870	47	10	751
747	239	47	10	761
759	452	47	10	773
778	862	53	10	797
792	852	53	10	811
802	643	53	10	821

811	543	53	10	829	
821	447	53	10	839	
835	321	53	10	853	
845	287	53	10	863	
860	12	53	10	877	
870	251	53	10	887	
891	30	53	10	907	
903	621	53	10	919	
913	555	53	10	929	
926	127	53	10	941	
938	400	53	10	953	
950	91	59	10	971	
963	916	59	10	983	
977	935	59	10	997	
989	691	59	10	1009	
1002	299	59	10	1021	
1020	282	59	10	1039	
1032	824	59	10	1051	
1050	536	59	11	1069	
1074	596	59	11	1093	
1085	28	59	11	1103	
1099	947	59	11	1117	
1111	162	59	11	1129	


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+-----+-----+-----+-----+-----+
| 1136 | 536 | 59 | 11 | 1153 |
+-----+-----+-----+-----+-----+
| 1152 | 1000 | 61 | 11 | 1171 |
+-----+-----+-----+-----+-----+
| 1169 | 251 | 61 | 11 | 1187 |
+-----+-----+-----+-----+-----+
| 1183 | 673 | 61 | 11 | 1201 |
+-----+-----+-----+-----+-----+
| 1205 | 559 | 61 | 11 | 1223 |
+-----+-----+-----+-----+-----+
| 1220 | 923 | 61 | 11 | 1237 |
+-----+-----+-----+-----+-----+
| 1236 | 81 | 67 | 11 | 1259 |
+-----+-----+-----+-----+-----+
| 1255 | 478 | 67 | 11 | 1277 |
+-----+-----+-----+-----+-----+
| 1269 | 198 | 67 | 11 | 1291 |
+-----+-----+-----+-----+-----+
| 1285 | 137 | 67 | 11 | 1307 |
+-----+-----+-----+-----+-----+
| 1306 | 75 | 67 | 11 | 1327 |
+-----+-----+-----+-----+-----+
| 1347 | 29 | 67 | 11 | 1367 |
+-----+-----+-----+-----+-----+
| 1361 | 231 | 67 | 11 | 1381 |
+-----+-----+-----+-----+-----+
| 1389 | 532 | 67 | 11 | 1409 |
+-----+-----+-----+-----+-----+
| 1404 | 58 | 67 | 11 | 1423 |
+-----+-----+-----+-----+-----+
| 1420 | 60 | 67 | 11 | 1439 |
+-----+-----+-----+-----+-----+
| 1436 | 964 | 71 | 11 | 1459 |
+-----+-----+-----+-----+-----+
| 1461 | 624 | 71 | 11 | 1483 |
+-----+-----+-----+-----+-----+
| 1477 | 502 | 71 | 11 | 1499 |
+-----+-----+-----+-----+-----+
| 1502 | 636 | 71 | 11 | 1523 |
+-----+-----+-----+-----+-----+
| 1522 | 986 | 71 | 11 | 1543 |
+-----+-----+-----+-----+-----+
| 1539 | 950 | 71 | 11 | 1559 |
+-----+-----+-----+-----+-----+
| 1561 | 735 | 73 | 11 | 1583 |
+-----+-----+-----+-----+-----+

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1579	866	73	11	1601
1600	203	73	11	1621
1616	83	73	11	1637
1649	14	73	11	1669
1673	522	79	11	1699
1698	226	79	11	1723
1716	282	79	11	1741
1734	88	79	11	1759
1759	636	79	11	1783
1777	860	79	11	1801
1800	324	79	11	1823
1824	424	79	11	1847
1844	999	79	11	1867
1863	682	83	11	1889
1887	814	83	11	1913
1906	979	83	11	1931
1926	538	83	11	1951
1954	278	83	11	1979
1979	580	83	11	2003
2005	773	83	11	2029
2040	911	89	11	2069
2070	506	89	11	2099
2103	628	89	11	2131

2125	282	89	11	2153
2152	309	89	11	2179
2195	858	89	11	2221
2217	442	89	11	2243
2247	654	89	11	2273
2278	82	97	11	2311
2315	428	97	11	2347
2339	442	97	11	2371
2367	283	97	11	2399
2392	538	97	11	2423
2416	189	97	11	2447
2447	438	97	11	2477
2473	912	97	11	2503
2502	1	97	11	2531
2528	167	97	11	2557
2565	272	97	11	2593
2601	209	101	11	2633
2640	927	101	11	2671
2668	386	101	11	2699
2701	653	101	11	2731
2737	669	101	11	2767
2772	431	101	11	2801
2802	793	103	11	2833


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+-----+-----+-----+-----+-----+
| 2831 | 588 | 103 | 11 | 2861 |
+-----+-----+-----+-----+-----+
| 2875 | 777 | 107 | 11 | 2909 |
+-----+-----+-----+-----+-----+
| 2906 | 939 | 107 | 11 | 2939 |
+-----+-----+-----+-----+-----+
| 2938 | 864 | 107 | 11 | 2971 |
+-----+-----+-----+-----+-----+
| 2979 | 627 | 107 | 11 | 3011 |
+-----+-----+-----+-----+-----+
| 3015 | 265 | 109 | 11 | 3049 |
+-----+-----+-----+-----+-----+
| 3056 | 976 | 109 | 11 | 3089 |
+-----+-----+-----+-----+-----+
| 3101 | 988 | 113 | 11 | 3137 |
+-----+-----+-----+-----+-----+
| 3151 | 507 | 113 | 11 | 3187 |
+-----+-----+-----+-----+-----+
| 3186 | 640 | 113 | 11 | 3221 |
+-----+-----+-----+-----+-----+
| 3224 | 15 | 113 | 11 | 3259 |
+-----+-----+-----+-----+-----+
| 3265 | 667 | 113 | 11 | 3299 |
+-----+-----+-----+-----+-----+
| 3299 | 24 | 127 | 11 | 3347 |
+-----+-----+-----+-----+-----+
| 3344 | 877 | 127 | 11 | 3391 |
+-----+-----+-----+-----+-----+
| 3387 | 240 | 127 | 11 | 3433 |
+-----+-----+-----+-----+-----+
| 3423 | 720 | 127 | 11 | 3469 |
+-----+-----+-----+-----+-----+
| 3466 | 93 | 127 | 11 | 3511 |
+-----+-----+-----+-----+-----+
| 3502 | 919 | 127 | 11 | 3547 |
+-----+-----+-----+-----+-----+
| 3539 | 635 | 127 | 11 | 3583 |
+-----+-----+-----+-----+-----+
| 3579 | 174 | 127 | 11 | 3623 |
+-----+-----+-----+-----+-----+
| 3616 | 647 | 127 | 11 | 3659 |
+-----+-----+-----+-----+-----+
| 3658 | 820 | 127 | 11 | 3701 |
+-----+-----+-----+-----+-----+
| 3697 | 56 | 127 | 11 | 3739 |
+-----+-----+-----+-----+-----+

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+-----+-----+-----+-----+-----+
| 3751 | 485 | 127 | 11 | 3793 |
+-----+-----+-----+-----+-----+
| 3792 | 210 | 127 | 11 | 3833 |
+-----+-----+-----+-----+-----+
| 3840 | 124 | 127 | 11 | 3881 |
+-----+-----+-----+-----+-----+
| 3883 | 546 | 127 | 11 | 3923 |
+-----+-----+-----+-----+-----+
| 3924 | 954 | 131 | 11 | 3967 |
+-----+-----+-----+-----+-----+
| 3970 | 262 | 131 | 11 | 4013 |
+-----+-----+-----+-----+-----+
| 4015 | 927 | 131 | 11 | 4057 |
+-----+-----+-----+-----+-----+
| 4069 | 957 | 131 | 11 | 4111 |
+-----+-----+-----+-----+-----+
| 4112 | 726 | 137 | 11 | 4159 |
+-----+-----+-----+-----+-----+
| 4165 | 583 | 137 | 11 | 4211 |
+-----+-----+-----+-----+-----+
| 4207 | 782 | 137 | 11 | 4253 |
+-----+-----+-----+-----+-----+
| 4252 | 37 | 137 | 11 | 4297 |
+-----+-----+-----+-----+-----+
| 4318 | 758 | 137 | 11 | 4363 |
+-----+-----+-----+-----+-----+
| 4365 | 777 | 137 | 11 | 4409 |
+-----+-----+-----+-----+-----+
| 4418 | 104 | 139 | 11 | 4463 |
+-----+-----+-----+-----+-----+
| 4468 | 476 | 139 | 11 | 4513 |
+-----+-----+-----+-----+-----+
| 4513 | 113 | 149 | 11 | 4567 |
+-----+-----+-----+-----+-----+
| 4567 | 313 | 149 | 11 | 4621 |
+-----+-----+-----+-----+-----+
| 4626 | 102 | 149 | 11 | 4679 |
+-----+-----+-----+-----+-----+
| 4681 | 501 | 149 | 11 | 4733 |
+-----+-----+-----+-----+-----+
| 4731 | 332 | 149 | 11 | 4783 |
+-----+-----+-----+-----+-----+
| 4780 | 786 | 149 | 11 | 4831 |
+-----+-----+-----+-----+-----+
| 4838 | 99 | 149 | 11 | 4889 |
+-----+-----+-----+-----+-----+

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+	-----+	-----+	-----+	-----+	-----+
	4901	658	149	11	4951
+	-----+	-----+	-----+	-----+	-----+
	4954	794	149	11	5003
+	-----+	-----+	-----+	-----+	-----+
	5008	37	151	11	5059
+	-----+	-----+	-----+	-----+	-----+
	5063	471	151	11	5113
+	-----+	-----+	-----+	-----+	-----+
	5116	94	157	11	5171
+	-----+	-----+	-----+	-----+	-----+
	5172	873	157	11	5227
+	-----+	-----+	-----+	-----+	-----+
	5225	918	157	11	5279
+	-----+	-----+	-----+	-----+	-----+
	5279	945	157	11	5333
+	-----+	-----+	-----+	-----+	-----+
	5334	211	157	11	5387
+	-----+	-----+	-----+	-----+	-----+
	5391	341	157	11	5443
+	-----+	-----+	-----+	-----+	-----+
	5449	11	163	11	5507
+	-----+	-----+	-----+	-----+	-----+
	5506	578	163	11	5563
+	-----+	-----+	-----+	-----+	-----+
	5566	494	163	11	5623
+	-----+	-----+	-----+	-----+	-----+
	5637	694	163	11	5693
+	-----+	-----+	-----+	-----+	-----+
	5694	252	163	11	5749
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	5763	451	167	11	5821
+	-----+	-----+	-----+	-----+	-----+
	5823	83	167	11	5881
+	-----+	-----+	-----+	-----+	-----+
	5896	689	167	11	5953
+	-----+	-----+	-----+	-----+	-----+
	5975	488	173	11	6037
+	-----+	-----+	-----+	-----+	-----+
	6039	214	173	11	6101
+	-----+	-----+	-----+	-----+	-----+
	6102	17	173	11	6163
+	-----+	-----+	-----+	-----+	-----+
	6169	469	173	11	6229
+	-----+	-----+	-----+	-----+	-----+
	6233	263	179	11	6299
+	-----+	-----+	-----+	-----+	-----+

6296	309	179	11	6361
6363	984	179	11	6427
6427	123	179	11	6491
6518	360	179	11	6581
6589	863	181	11	6653
6655	122	181	11	6719
6730	522	191	11	6803
6799	539	191	11	6871
6878	181	191	11	6949
6956	64	191	11	7027
7033	387	191	11	7103
7108	967	191	11	7177
7185	843	191	11	7253
7281	999	193	11	7351
7360	76	197	11	7433
7445	142	197	11	7517
7520	599	197	11	7591
7596	576	199	11	7669
7675	176	211	11	7759
7770	392	211	11	7853
7855	332	211	11	7937
7935	291	211	11	8017
8030	913	211	11	8111


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+-----+-----+-----+-----+-----+
| 8111 | 608 | 211 | 11 | 8191 |
+-----+-----+-----+-----+-----+
| 8194 | 212 | 211 | 11 | 8273 |
+-----+-----+-----+-----+-----+
| 8290 | 696 | 211 | 11 | 8369 |
+-----+-----+-----+-----+-----+
| 8377 | 931 | 223 | 11 | 8467 |
+-----+-----+-----+-----+-----+
| 8474 | 326 | 223 | 11 | 8563 |
+-----+-----+-----+-----+-----+
| 8559 | 228 | 223 | 11 | 8647 |
+-----+-----+-----+-----+-----+
| 8654 | 706 | 223 | 11 | 8741 |
+-----+-----+-----+-----+-----+
| 8744 | 144 | 223 | 11 | 8831 |
+-----+-----+-----+-----+-----+
| 8837 | 83 | 223 | 11 | 8923 |
+-----+-----+-----+-----+-----+
| 8928 | 743 | 223 | 11 | 9013 |
+-----+-----+-----+-----+-----+
| 9019 | 187 | 223 | 11 | 9103 |
+-----+-----+-----+-----+-----+
| 9111 | 654 | 227 | 11 | 9199 |
+-----+-----+-----+-----+-----+
| 9206 | 359 | 227 | 11 | 9293 |
+-----+-----+-----+-----+-----+
| 9303 | 493 | 229 | 11 | 9391 |
+-----+-----+-----+-----+-----+
| 9400 | 369 | 233 | 11 | 9491 |
+-----+-----+-----+-----+-----+
| 9497 | 981 | 233 | 11 | 9587 |
+-----+-----+-----+-----+-----+
| 9601 | 276 | 239 | 11 | 9697 |
+-----+-----+-----+-----+-----+
| 9708 | 647 | 239 | 11 | 9803 |
+-----+-----+-----+-----+-----+
| 9813 | 389 | 239 | 11 | 9907 |
+-----+-----+-----+-----+-----+
| 9916 | 80 | 239 | 11 | 10009 |
+-----+-----+-----+-----+-----+
| 10017 | 396 | 241 | 11 | 10111 |
+-----+-----+-----+-----+-----+
| 10120 | 580 | 251 | 11 | 10223 |
+-----+-----+-----+-----+-----+
| 10241 | 873 | 251 | 11 | 10343 |
+-----+-----+-----+-----+-----+

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+-----+-----+-----+-----+-----+
| 10351 | 15   | 251  | 11   | 10453 |
+-----+-----+-----+-----+-----+
| 10458 | 976  | 251  | 11   | 10559 |
+-----+-----+-----+-----+-----+
| 10567 | 584  | 251  | 11   | 10667 |
+-----+-----+-----+-----+-----+
| 10676 | 267  | 257  | 11   | 10781 |
+-----+-----+-----+-----+-----+
| 10787 | 876  | 257  | 11   | 10891 |
+-----+-----+-----+-----+-----+
| 10899 | 642  | 257  | 12   | 11003 |
+-----+-----+-----+-----+-----+
| 11015 | 794  | 257  | 12   | 11119 |
+-----+-----+-----+-----+-----+
| 11130 | 78   | 263  | 12   | 11239 |
+-----+-----+-----+-----+-----+
| 11245 | 736  | 263  | 12   | 11353 |
+-----+-----+-----+-----+-----+
| 11358 | 882  | 269  | 12   | 11471 |
+-----+-----+-----+-----+-----+
| 11475 | 251  | 269  | 12   | 11587 |
+-----+-----+-----+-----+-----+
| 11590 | 434  | 269  | 12   | 11701 |
+-----+-----+-----+-----+-----+
| 11711 | 204  | 269  | 12   | 11821 |
+-----+-----+-----+-----+-----+
| 11829 | 256  | 271  | 12   | 11941 |
+-----+-----+-----+-----+-----+
| 11956 | 106  | 277  | 12   | 12073 |
+-----+-----+-----+-----+-----+
| 12087 | 375  | 277  | 12   | 12203 |
+-----+-----+-----+-----+-----+
| 12208 | 148  | 277  | 12   | 12323 |
+-----+-----+-----+-----+-----+
| 12333 | 496  | 281  | 12   | 12451 |
+-----+-----+-----+-----+-----+
| 12460 | 88   | 281  | 12   | 12577 |
+-----+-----+-----+-----+-----+
| 12593 | 826  | 293  | 12   | 12721 |
+-----+-----+-----+-----+-----+
| 12726 | 71   | 293  | 12   | 12853 |
+-----+-----+-----+-----+-----+
| 12857 | 925  | 293  | 12   | 12983 |
+-----+-----+-----+-----+-----+
| 13002 | 760  | 293  | 12   | 13127 |
+-----+-----+-----+-----+-----+

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+-----+-----+-----+-----+-----+
| 13143 | 130  | 293  | 12   | 13267 |
+-----+-----+-----+-----+-----+
| 13284 | 641   | 307   | 12   | 13421 |
+-----+-----+-----+-----+-----+
| 13417 | 400   | 307   | 12   | 13553 |
+-----+-----+-----+-----+-----+
| 13558 | 480   | 307   | 12   | 13693 |
+-----+-----+-----+-----+-----+
| 13695 | 76    | 307   | 12   | 13829 |
+-----+-----+-----+-----+-----+
| 13833 | 665   | 307   | 12   | 13967 |
+-----+-----+-----+-----+-----+
| 13974 | 910   | 307   | 12   | 14107 |
+-----+-----+-----+-----+-----+
| 14115 | 467   | 311   | 12   | 14251 |
+-----+-----+-----+-----+-----+
| 14272 | 964   | 311   | 12   | 14407 |
+-----+-----+-----+-----+-----+
| 14415 | 625   | 313   | 12   | 14551 |
+-----+-----+-----+-----+-----+
| 14560 | 362   | 317   | 12   | 14699 |
+-----+-----+-----+-----+-----+
| 14713 | 759   | 317   | 12   | 14851 |
+-----+-----+-----+-----+-----+
| 14862 | 728   | 331   | 12   | 15013 |
+-----+-----+-----+-----+-----+
| 15011 | 343   | 331   | 12   | 15161 |
+-----+-----+-----+-----+-----+
| 15170 | 113   | 331   | 12   | 15319 |
+-----+-----+-----+-----+-----+
| 15325 | 137   | 331   | 12   | 15473 |
+-----+-----+-----+-----+-----+
| 15496 | 308   | 331   | 12   | 15643 |
+-----+-----+-----+-----+-----+
| 15651 | 800   | 337   | 12   | 15803 |
+-----+-----+-----+-----+-----+
| 15808 | 177   | 337   | 12   | 15959 |
+-----+-----+-----+-----+-----+
| 15977 | 961   | 337   | 12   | 16127 |
+-----+-----+-----+-----+-----+
| 16161 | 958   | 347   | 12   | 16319 |
+-----+-----+-----+-----+-----+
| 16336 | 72    | 347   | 12   | 16493 |
+-----+-----+-----+-----+-----+
| 16505 | 732   | 347   | 12   | 16661 |
+-----+-----+-----+-----+-----+

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+-----+-----+-----+-----+-----+
| 16674 | 145   | 349   | 12    | 16831 |
+-----+-----+-----+-----+-----+
| 16851 | 577   | 353   | 12    | 17011 |
+-----+-----+-----+-----+-----+
| 17024 | 305   | 353   | 12    | 17183 |
+-----+-----+-----+-----+-----+
| 17195 | 50    | 359   | 12    | 17359 |
+-----+-----+-----+-----+-----+
| 17376 | 351   | 359   | 12    | 17539 |
+-----+-----+-----+-----+-----+
| 17559 | 175   | 367   | 12    | 17729 |
+-----+-----+-----+-----+-----+
| 17742 | 727   | 367   | 12    | 17911 |
+-----+-----+-----+-----+-----+
| 17929 | 902   | 367   | 12    | 18097 |
+-----+-----+-----+-----+-----+
| 18116 | 409   | 373   | 12    | 18289 |
+-----+-----+-----+-----+-----+
| 18309 | 776   | 373   | 12    | 18481 |
+-----+-----+-----+-----+-----+
| 18503 | 586   | 379   | 12    | 18679 |
+-----+-----+-----+-----+-----+
| 18694 | 451   | 379   | 12    | 18869 |
+-----+-----+-----+-----+-----+
| 18909 | 287   | 383   | 12    | 19087 |
+-----+-----+-----+-----+-----+
| 19126 | 246   | 389   | 12    | 19309 |
+-----+-----+-----+-----+-----+
| 19325 | 222   | 389   | 12    | 19507 |
+-----+-----+-----+-----+-----+
| 19539 | 563   | 397   | 12    | 19727 |
+-----+-----+-----+-----+-----+
| 19740 | 839   | 397   | 12    | 19927 |
+-----+-----+-----+-----+-----+
| 19939 | 897   | 401   | 12    | 20129 |
+-----+-----+-----+-----+-----+
| 20152 | 409   | 401   | 12    | 20341 |
+-----+-----+-----+-----+-----+
| 20355 | 618   | 409   | 12    | 20551 |
+-----+-----+-----+-----+-----+
| 20564 | 439   | 409   | 12    | 20759 |
+-----+-----+-----+-----+-----+
| 20778 | 95    | 419   | 13    | 20983 |
+-----+-----+-----+-----+-----+
| 20988 | 448   | 419   | 13    | 21191 |
+-----+-----+-----+-----+-----+

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+-----+-----+-----+-----+-----+
| 21199 | 133  | 419  | 13   | 21401 |
+-----+-----+-----+-----+-----+
| 21412 | 938   | 419  | 13   | 21613 |
+-----+-----+-----+-----+-----+
| 21629 | 423   | 431  | 13   | 21841 |
+-----+-----+-----+-----+-----+
| 21852 | 90     | 431  | 13   | 22063 |
+-----+-----+-----+-----+-----+
| 22073 | 640   | 431  | 13   | 22283 |
+-----+-----+-----+-----+-----+
| 22301 | 922   | 433  | 13   | 22511 |
+-----+-----+-----+-----+-----+
| 22536 | 250   | 439  | 13   | 22751 |
+-----+-----+-----+-----+-----+
| 22779 | 367   | 439  | 13   | 22993 |
+-----+-----+-----+-----+-----+
| 23010 | 447   | 443  | 13   | 23227 |
+-----+-----+-----+-----+-----+
| 23252 | 559   | 449  | 13   | 23473 |
+-----+-----+-----+-----+-----+
| 23491 | 121   | 457  | 13   | 23719 |
+-----+-----+-----+-----+-----+
| 23730 | 623   | 457  | 13   | 23957 |
+-----+-----+-----+-----+-----+
| 23971 | 450   | 457  | 13   | 24197 |
+-----+-----+-----+-----+-----+
| 24215 | 253   | 461  | 13   | 24443 |
+-----+-----+-----+-----+-----+
| 24476 | 106   | 467  | 13   | 24709 |
+-----+-----+-----+-----+-----+
| 24721 | 863   | 467  | 13   | 24953 |
+-----+-----+-----+-----+-----+
| 24976 | 148   | 479  | 13   | 25219 |
+-----+-----+-----+-----+-----+
| 25230 | 427   | 479  | 13   | 25471 |
+-----+-----+-----+-----+-----+
| 25493 | 138   | 479  | 13   | 25733 |
+-----+-----+-----+-----+-----+
| 25756 | 794   | 487  | 13   | 26003 |
+-----+-----+-----+-----+-----+
| 26022 | 247   | 487  | 13   | 26267 |
+-----+-----+-----+-----+-----+
| 26291 | 562   | 491  | 13   | 26539 |
+-----+-----+-----+-----+-----+
| 26566 | 53    | 499  | 13   | 26821 |
+-----+-----+-----+-----+-----+

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+-----+-----+-----+-----+-----+
| 26838 | 135  | 499  | 13   | 27091 |
+-----+-----+-----+-----+-----+
| 27111 | 21    | 503  | 13   | 27367 |
+-----+-----+-----+-----+-----+
| 27392 | 201   | 509  | 13   | 27653 |
+-----+-----+-----+-----+-----+
| 27682 | 169   | 521  | 13   | 27953 |
+-----+-----+-----+-----+-----+
| 27959 | 70    | 521  | 13   | 28229 |
+-----+-----+-----+-----+-----+
| 28248 | 386   | 521  | 13   | 28517 |
+-----+-----+-----+-----+-----+
| 28548 | 226   | 523  | 13   | 28817 |
+-----+-----+-----+-----+-----+
| 28845 | 3      | 541  | 13   | 29131 |
+-----+-----+-----+-----+-----+
| 29138 | 769   | 541  | 13   | 29423 |
+-----+-----+-----+-----+-----+
| 29434 | 590   | 541  | 13   | 29717 |
+-----+-----+-----+-----+-----+
| 29731 | 672   | 541  | 13   | 30013 |
+-----+-----+-----+-----+-----+
| 30037 | 713   | 547  | 13   | 30323 |
+-----+-----+-----+-----+-----+
| 30346 | 967   | 547  | 13   | 30631 |
+-----+-----+-----+-----+-----+
| 30654 | 368   | 557  | 14   | 30949 |
+-----+-----+-----+-----+-----+
| 30974 | 348   | 557  | 14   | 31267 |
+-----+-----+-----+-----+-----+
| 31285 | 119   | 563  | 14   | 31583 |
+-----+-----+-----+-----+-----+
| 31605 | 503   | 569  | 14   | 31907 |
+-----+-----+-----+-----+-----+
| 31948 | 181   | 571  | 14   | 32251 |
+-----+-----+-----+-----+-----+
| 32272 | 394   | 577  | 14   | 32579 |
+-----+-----+-----+-----+-----+
| 32601 | 189   | 587  | 14   | 32917 |
+-----+-----+-----+-----+-----+
| 32932 | 210   | 587  | 14   | 33247 |
+-----+-----+-----+-----+-----+
| 33282 | 62    | 593  | 14   | 33601 |
+-----+-----+-----+-----+-----+
| 33623 | 273   | 593  | 14   | 33941 |
+-----+-----+-----+-----+-----+

```



```
+-----+-----+-----+-----+-----+
| 33961 | 554   | 599   | 14    | 34283 |
+-----+-----+-----+-----+-----+
| 34302 | 936   | 607   | 14    | 34631 |
+-----+-----+-----+-----+-----+
| 34654 | 483   | 607   | 14    | 34981 |
+-----+-----+-----+-----+-----+
| 35031 | 397   | 613   | 14    | 35363 |
+-----+-----+-----+-----+-----+
| 35395 | 241   | 619   | 14    | 35731 |
+-----+-----+-----+-----+-----+
| 35750 | 500   | 631   | 14    | 36097 |
+-----+-----+-----+-----+-----+
| 36112 | 12    | 631   | 14    | 36457 |
+-----+-----+-----+-----+-----+
| 36479 | 958   | 641   | 14    | 36833 |
+-----+-----+-----+-----+-----+
| 36849 | 524   | 641   | 14    | 37201 |
+-----+-----+-----+-----+-----+
| 37227 | 8     | 643   | 14    | 37579 |
+-----+-----+-----+-----+-----+
| 37606 | 100   | 653   | 14    | 37967 |
+-----+-----+-----+-----+-----+
| 37992 | 339   | 653   | 14    | 38351 |
+-----+-----+-----+-----+-----+
| 38385 | 804   | 659   | 14    | 38749 |
+-----+-----+-----+-----+-----+
| 38787 | 510   | 673   | 14    | 39163 |
+-----+-----+-----+-----+-----+
| 39176 | 18    | 673   | 14    | 39551 |
+-----+-----+-----+-----+-----+
| 39576 | 412   | 677   | 14    | 39953 |
+-----+-----+-----+-----+-----+
| 39980 | 394   | 683   | 14    | 40361 |
+-----+-----+-----+-----+-----+
| 40398 | 830   | 691   | 15    | 40787 |
+-----+-----+-----+-----+-----+
| 40816 | 535   | 701   | 15    | 41213 |
+-----+-----+-----+-----+-----+
| 41226 | 199   | 701   | 15    | 41621 |
+-----+-----+-----+-----+-----+
| 41641 | 27    | 709   | 15    | 42043 |
+-----+-----+-----+-----+-----+
| 42067 | 298   | 709   | 15    | 42467 |
+-----+-----+-----+-----+-----+
| 42490 | 368   | 719   | 15    | 42899 |
+-----+-----+-----+-----+-----+
```



```

+-----+-----+-----+-----+-----+
| 42916 | 755   | 727   | 15    | 43331 |
+-----+-----+-----+-----+-----+
| 43388 | 379   | 727   | 15    | 43801 |
+-----+-----+-----+-----+-----+
| 43840 | 73    | 733   | 15    | 44257 |
+-----+-----+-----+-----+-----+
| 44279 | 387   | 739   | 15    | 44701 |
+-----+-----+-----+-----+-----+
| 44729 | 457   | 751   | 15    | 45161 |
+-----+-----+-----+-----+-----+
| 45183 | 761   | 751   | 15    | 45613 |
+-----+-----+-----+-----+-----+
| 45638 | 855   | 757   | 15    | 46073 |
+-----+-----+-----+-----+-----+
| 46104 | 370   | 769   | 15    | 46549 |
+-----+-----+-----+-----+-----+
| 46574 | 261   | 769   | 15    | 47017 |
+-----+-----+-----+-----+-----+
| 47047 | 299   | 787   | 15    | 47507 |
+-----+-----+-----+-----+-----+
| 47523 | 920   | 787   | 15    | 47981 |
+-----+-----+-----+-----+-----+
| 48007 | 269   | 787   | 15    | 48463 |
+-----+-----+-----+-----+-----+
| 48489 | 862   | 797   | 15    | 48953 |
+-----+-----+-----+-----+-----+
| 48976 | 349   | 809   | 15    | 49451 |
+-----+-----+-----+-----+-----+
| 49470 | 103   | 809   | 15    | 49943 |
+-----+-----+-----+-----+-----+
| 49978 | 115   | 821   | 15    | 50461 |
+-----+-----+-----+-----+-----+
| 50511 | 93    | 821   | 16    | 50993 |
+-----+-----+-----+-----+-----+
| 51017 | 982   | 827   | 16    | 51503 |
+-----+-----+-----+-----+-----+
| 51530 | 432   | 839   | 16    | 52027 |
+-----+-----+-----+-----+-----+
| 52062 | 340   | 853   | 16    | 52571 |
+-----+-----+-----+-----+-----+
| 52586 | 173   | 853   | 16    | 53093 |
+-----+-----+-----+-----+-----+
| 53114 | 421   | 857   | 16    | 53623 |
+-----+-----+-----+-----+-----+
| 53650 | 330   | 863   | 16    | 54163 |
+-----+-----+-----+-----+-----+

```


+-----+	+-----+	+-----+	+-----+	+-----+
54188	624	877	16	54713
+-----+	+-----+	+-----+	+-----+	+-----+
54735	233	877	16	55259
+-----+	+-----+	+-----+	+-----+	+-----+
55289	362	883	16	55817
+-----+	+-----+	+-----+	+-----+	+-----+
55843	963	907	16	56393
+-----+	+-----+	+-----+	+-----+	+-----+
56403	471	907	16	56951
+-----+	+-----+	+-----+	+-----+	+-----+

Table 2: Systematic Indices and Other Parameters

5.7. Operating with Octets, Symbols, and Matrices

5.7.1. General

The remainder of this section describes the arithmetic operations that are used to generate encoding symbols from source symbols and to generate source symbols from encoding symbols. Mathematically, octets can be thought of as elements of a finite field, i.e., the finite field GF(256) with 256 elements, and thus the addition and multiplication operations and identity elements and inverses over both operations are defined. Matrix operations and symbol operations are defined based on the arithmetic operations on octets. This allows a full implementation of these arithmetic operations without having to understand the underlying mathematics of finite fields.

5.7.2. Arithmetic Operations on Octets

Octets are mapped to non-negative integers in the range 0 through 255 in the usual way: A single octet of data from a symbol, $B[7], B[6], B[5], B[4], B[3], B[2], B[1], B[0]$, where $B[7]$ is the highest order bit and $B[0]$ is the lowest order bit, is mapped to the integer $i = B[7] * 128 + B[6] * 64 + B[5] * 32 + B[4] * 16 + B[3] * 8 + B[2] * 4 + B[1] * 2 + B[0]$.

The addition of two octets u and v is defined as the exclusive-or operation, i.e.,

$$u + v = u \wedge v.$$

Subtraction is defined in the same way, so we also have

$$u - v = u \wedge v.$$

The zero element (additive identity) is the octet represented by the integer 0. The additive inverse of u is simply u , i.e.,

$$u + u = 0.$$

The multiplication of two octets is defined with the help of two tables OCT_EXP and OCT_LOG, which are given in [Section 5.7.3](#) and [Section 5.7.4](#), respectively. The table OCT_LOG maps octets (other than the zero element) to non-negative integers, and OCT_EXP maps non-negative integers to octets. For two octets u and v , we define

$$u * v = \begin{cases} 0, & \text{if either } u \text{ or } v \text{ are } 0, \\ \text{OCT_EXP}[\text{OCT_LOG}[u] + \text{OCT_LOG}[v]] & \text{otherwise.} \end{cases}$$

Note that the '+' on the right-hand side of the above is the usual integer addition, since its arguments are ordinary integers.

The division u / v of two octets u and v , and where $v \neq 0$, is defined as follows:

$$u / v = \begin{cases} 0, & \text{if } u == 0, \\ \text{OCT_EXP}[\text{OCT_LOG}[u] - \text{OCT_LOG}[v] + 255] & \text{otherwise.} \end{cases}$$

The one element (multiplicative identity) is the octet represented by the integer 1. For an octet u that is not the zero element, i.e., the multiplicative inverse of u is

$$\text{OCT_EXP}[255 - \text{OCT_LOG}[u]].$$

The octet denoted by α is the octet with the integer representation 2. If i is a non-negative integer $0 \leq i < 256$, we have

$$\alpha^i = \text{OCT_EXP}[i].$$

5.7.3. The Table OCT_EXP

The table OCT_EXP contains 510 octets. The indexing starts at 0 and ranges to 509, and the entries are the octets with the following positive integer representation:

1, 2, 4, 8, 16, 32, 64, 128, 29, 58, 116, 232, 205, 135, 19, 38, 76, 152, 45, 90, 180, 117, 234, 201, 143, 3, 6, 12, 24, 48, 96, 192, 157, 39, 78, 156, 37, 74, 148, 53, 106, 212, 181, 119, 238, 193, 159, 35, 70, 140, 5, 10, 20, 40, 80, 160, 93, 186, 105, 210, 185, 111, 222, 161, 95, 190, 97, 194, 153, 47, 94, 188, 101, 202, 137, 15, 30, 60, 120, 240, 253, 231, 211, 187, 107, 214, 177, 127, 254, 225, 223, 163, 91, 182, 113, 226, 217, 175, 67, 134, 17, 34, 68, 136, 13, 26, 52, 104, 208, 189, 103, 206, 129, 31, 62, 124, 248, 237, 199, 147, 59, 118, 236, 197, 151, 51, 102, 204, 133, 23, 46, 92, 184, 109, 218, 169, 79, 158, 33, 66, 132, 21, 42, 84, 168, 77, 154, 41, 82, 164, 85, 170, 73, 146, 57, 114, 228, 213, 183, 115, 230, 209, 191, 99, 198, 145, 63, 126, 252, 229, 215, 179, 123, 246, 241, 255, 227, 219, 171, 75, 150, 49, 98, 196, 149, 55, 110, 220, 165, 87, 174, 65, 130, 25, 50, 100, 200, 141, 7, 14, 28, 56, 112, 224, 221, 167, 83, 166, 81, 162, 89, 178, 121, 242, 249, 239, 195, 155, 43, 86, 172, 69, 138, 9, 18, 36, 72, 144, 61, 122, 244, 245, 247, 243, 251, 235, 203, 139, 11, 22, 44, 88, 176, 125, 250, 233, 207, 131, 27, 54, 108, 216, 173, 71, 142, 1, 2, 4, 8, 16, 32, 64, 128, 29, 58, 116, 232, 205, 135, 19, 38, 76, 152, 45, 90, 180, 117, 234, 201, 143, 3, 6, 12, 24, 48, 96, 192, 157, 39, 78, 156, 37, 74, 148, 53, 106, 212, 181, 119, 238, 193, 159, 35, 70, 140, 5, 10, 20, 40, 80, 160, 93, 186, 105, 210, 185, 111, 222, 161, 95, 190, 97, 194, 153, 47, 94, 188, 101, 202, 137, 15, 30, 60, 120, 240, 253, 231, 211, 187, 107, 214, 177, 127, 254, 225, 223, 163, 91, 182, 113, 226, 217, 175, 67, 134, 17, 34, 68, 136, 13, 26, 52, 104, 208, 189, 103, 206, 129, 31, 62, 124, 248, 237, 199, 147, 59, 118, 236, 197, 151, 51, 102, 204, 133, 23, 46, 92, 184, 109, 218, 169, 79, 158, 33, 66, 132, 21, 42, 84, 168, 77, 154, 41, 82, 164, 85, 170, 73, 146, 57, 114, 228, 213, 183, 115, 230, 209, 191, 99, 198, 145, 63, 126, 252, 229, 215, 179, 123, 246, 241, 255, 227, 219, 171, 75, 150, 49, 98, 196, 149, 55, 110, 220, 165, 87, 174, 65, 130, 25, 50, 100, 200, 141, 7, 14, 28, 56, 112, 224, 221, 167, 83, 166, 81, 162, 89, 178, 121, 242, 249, 239, 195, 155, 43, 86, 172, 69, 138, 9, 18, 36, 72, 144, 61, 122, 244, 245, 247, 243, 251, 235, 203, 139, 11, 22, 44, 88, 176, 125, 250, 233, 207, 131, 27, 54, 108, 216, 173, 71, 142

5.7.4. The Table OCT_LOG

The table OCT_LOG contains 255 non-negative integers. The table is indexed by octets interpreted as integers. The octet corresponding to the zero element, which is represented by the integer 0, is excluded as an index, and thus indexing starts at 1 and ranges up to 255, and the entries are the following:

0, 1, 25, 2, 50, 26, 198, 3, 223, 51, 238, 27, 104, 199, 75, 4, 100, 224, 14, 52, 141, 239, 129, 28, 193, 105, 248, 200, 8, 76, 113, 5, 138, 101, 47, 225, 36, 15, 33, 53, 147, 142, 218, 240, 18, 130, 69, 29, 181, 194, 125, 106, 39, 249, 185, 201, 154, 9, 120, 77, 228, 114,

166, 6, 191, 139, 98, 102, 221, 48, 253, 226, 152, 37, 179, 16, 145,
 34, 136, 54, 208, 148, 206, 143, 150, 219, 189, 241, 210, 19, 92,
 131, 56, 70, 64, 30, 66, 182, 163, 195, 72, 126, 110, 107, 58, 40,
 84, 250, 133, 186, 61, 202, 94, 155, 159, 10, 21, 121, 43, 78, 212,
 229, 172, 115, 243, 167, 87, 7, 112, 192, 247, 140, 128, 99, 13, 103,
 74, 222, 237, 49, 197, 254, 24, 227, 165, 153, 119, 38, 184, 180,
 124, 17, 68, 146, 217, 35, 32, 137, 46, 55, 63, 209, 91, 149, 188,
 207, 205, 144, 135, 151, 178, 220, 252, 190, 97, 242, 86, 211, 171,
 20, 42, 93, 158, 132, 60, 57, 83, 71, 109, 65, 162, 31, 45, 67, 216,
 183, 123, 164, 118, 196, 23, 73, 236, 127, 12, 111, 246, 108, 161,
 59, 82, 41, 157, 85, 170, 251, 96, 134, 177, 187, 204, 62, 90, 203,
 89, 95, 176, 156, 169, 160, 81, 11, 245, 22, 235, 122, 117, 44, 215,
 79, 174, 213, 233, 230, 231, 173, 232, 116, 214, 244, 234, 168, 80,
 88, 175

5.7.5. Operations on Symbols

Operations on symbols have the same semantics as operations on vectors of octets of length T in this specification. Thus, if U and V are two symbols formed by the octets $u[0], \dots, u[T-1]$ and $v[0], \dots, v[T-1]$, respectively, the sum of symbols $U + V$ is defined to be the component-wise sum of octets, i.e., equal to the symbol D formed by the octets $d[0], \dots, d[T-1]$, such that

$$d[i] = u[i] + v[i], \quad 0 \leq i < T.$$

Furthermore, if β is an octet, the product $\beta * U$ is defined to be the symbol D obtained by multiplying each octet of U by β , i.e.,

$$d[i] = \beta * u[i], \quad 0 \leq i < T.$$

5.7.6. Operations on Matrices

All matrices in this specification have entries that are octets, and thus matrix operations and definitions are defined in terms of the underlying octet arithmetic, e.g., operations on a matrix, matrix rank, and matrix inversion.

5.8. Requirements for a Compliant Decoder

If a RaptorQ-compliant decoder receives a mathematically sufficient set of encoding symbols generated according to the encoder specification in [Section 5.3](#) for reconstruction of a source block, then such a decoder SHOULD recover the entire source block.

A RaptorQ-compliant decoder SHALL have the following recovery properties for source blocks with K' source symbols for all values of K' in Table 2 of [Section 5.6](#).

1. If the decoder receives K' encoding symbols generated according to the encoder specification in [Section 5.3](#) with corresponding ESIs chosen independently and uniformly at random from the range of possible ESIs, then on average the decoder will fail to recover the entire source block at most 1 out of 100 times.
2. If the decoder receives $K'+1$ encoding symbols generated according to the encoder specification in [Section 5.3](#) with corresponding ESIs chosen independently and uniformly at random from the range of possible ESIs, then on average the decoder will fail to recover the entire source block at most 1 out of 10,000 times.
3. If the decoder receives $K'+2$ encoding symbols generated according to the encoder specification in [Section 5.3](#) with corresponding ESIs chosen independently and uniformly at random from the range of possible ESIs, then on average the decoder will fail to recover the entire source block at most 1 out of 1,000,000 times.

Note that the Example FEC Decoder specified in [Section 5.4](#) fulfills both requirements, i.e.,

1. it can reconstruct a source block as long as it receives a mathematically sufficient set of encoding symbols generated according to the encoder specification in [Section 5.3](#), and
2. it fulfills the mandatory recovery properties from above.

6. Security Considerations

Data delivery can be subject to denial-of-service attacks by attackers that send corrupted packets that are accepted as legitimate by receivers. This is particularly a concern for multicast delivery because a corrupted packet may be injected into the session close to the root of the multicast tree, in which case the corrupted packet will arrive at many receivers. The use of even one corrupted packet containing encoding data may result in the decoding of an object that is completely corrupted and unusable. It is thus RECOMMENDED that source authentication and integrity checking are applied to decoded objects before delivering objects to an application. For example, a SHA-256 hash [[FIPS.180-3.2008](#)] of an object may be appended before transmission, and the SHA-256 hash is computed and checked after the object is decoded but before it is delivered to an application. Source authentication SHOULD be provided, for example, by including a digital signature verifiable by the receiver computed on top of the hash value. It is also RECOMMENDED that a packet authentication protocol such as TESLA [[RFC4082](#)] be used to detect and discard corrupted packets upon arrival. This method may also be used to provide source authentication. Furthermore, it is RECOMMENDED that

Reverse Path Forwarding checks be enabled in all network routers and switches along the path from the sender to receivers to limit the possibility of a bad agent successfully injecting a corrupted packet into the multicast tree data path.

Another security concern is that some FEC information may be obtained by receivers out-of-band in a session description, and if the session description is forged or corrupted, then the receivers will not use the correct protocol for decoding content from received packets. To avoid these problems, it is RECOMMENDED that measures be taken to prevent receivers from accepting incorrect session descriptions, e.g., by using source authentication to ensure that receivers only accept legitimate session descriptions from authorized senders.

7. IANA Considerations

Values of FEC Encoding IDs and FEC Instance IDs are subject to IANA registration. For general guidelines on IANA considerations as they apply to this document, see [RFC5052]. IANA has assigned the value 6 under the ietf:rmt:fec:encoding registry to "RaptorQ Code" as the Fully-Specified FEC Encoding ID value associated with this specification.

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