Approaches to the problem of making PAKEs quantum -safe

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The problem

All existing “industry” PAKE protocols are quantum-insecure:

Underlying hard problems (DLP, ECDLP and factoring)

can be solved on quantum computer in polynomial time by Shor’s algorithm.

Example

Shor's algorithm for ECDLP: space ~ 6n qubits , time ~ $360n^3$ (John Proos and Christof Zalka, 2003)

Most popular curves:

ed25519 (Edwards curve)

secp256k1 (Bitcoin, Ethereum)

P-256 (NIST standard)

space ~ 6*256 = 1536 qubits , time ~ $360*256^3$ operations

Isogeny basics

E_1 , E_2 - elliptic curves over F_q

Isogeny E_1 -> E_2:

$\varphi(x, y)$ = $(\frac{f_1(x,y)}{f_2(x,y)}, \frac{g_1(x,y)}{g_2(x,y)})$

$\varphi(\infty)$ = ∞

(equivalently, $\varphi(P + Q) = \varphi(P) + \varphi(Q)$)

(f_1, f_2, g_1, g_2 are polynomials)

Degree of isogeny φ is max degree of $f_1(x, y)$ and $f_2(x, y)$

Example

$E_1 : y^2 = x^3 + x + 1$ and $E_2 : y^2 = x^3 + 4x + 13$ over F_{19}

$$(x, y) = \left(\frac{x^3 - 4x^2 - 8x - 8}{(x-2)^2}, y\frac{x^3 - 6x^2 + 5x - 6}{(x-2)^3}\right)$$

deg φ = 3

A = (9, 6) , B = (14, 2) and $C = A + B$ = (5, 6)

φ (9, 6) = (14, 1)
φ (14, 2) = (17, 4)
φ (5, 6) = (8, 5)

Group homomorphism: φ (9, 6) + φ (14, 2) = φ (5, 6)

Construction of isogenies

Isogeny is a group homomorphism:

ker φ = { $P \in E$: $\varphi(P) = \infty$ }

Let's K is some subgroup of E,

exists φ_K : E -> E/K such that ker φ_K is K *and deg* φ_K = | K |

Isogeny can be calculated by Velu's algorithm (1971) :

Input : curve E_1 , K

Output: curve E_2, map φ

Construction of isogenies

Another way to express isogeny E -> E/K :

E -> $E/$<G_K>

where G_K is generator of kernel group K

Tate's theorem:

Two curves E_1 , E_2 are isogenous over F_q if and only if $\#E_1 = \#E_2$

Example

$E_1: y^2 = x^3 + x + 1$ and $E_2: y^2 = x^3 + 4x + 13$
over field F_{19}, $\#E_1 = \#E_2 = 21$

$$\varphi(x, y) = \left(\frac{x^3 - 4x^2 - 8x - 8}{x^2 - 4x + 4}, \frac{x^3y - 6x^2y + 5xy - 6y}{x^3 - 6x^2 - 7x - 8}\right)$$

deg $\varphi = 3$
Kernel of isogeny is a subgroup K = {∞, (2, 7), (2, 12)}
Kernel's generators are (2, 7), (2, 12),
so denote G_K = (2, 7) or G_K = (2, 12) $E_2 = E_1/\langle G_K \rangle$

Hard problem

Given E_1 , E_2 - elliptic curves over F_q, $\#E_1 = \#E_2$

Find isogeny φ *between* E_1 and E_2

n -torsion subgroup

$E[n]$ = { $R \in E(\overline{F_q})$: $n * R = \infty$ }

$E[n]$ is isomorphic to $\mathrm{Z/nZ} \times \mathrm{Z/nZ}$ $\mathrm{(i.e.\ has\ order = n^2}$)
$\mathrm{if\ gcd}\ (n, q) = 1$

Base points P and $Q \in E[n]$: each $C \in E[n]$ can be expressed as
C = $x * P + y * Q$
where x, y ∈ [0, n)

Supersingular curve

#E(GF(p^n)) = $p^n + 1 - t$, *where* t – trace of Frobenius

if t == 0 mod p :

    E is supersingular

else

    E is ordinary

SIDH (Supersingular Isogeny Diffie-Hellman)
D. Jao and L. De Feo, 2011

supersingular curve over F_{p^2} that contains subgroups E[2^{e2}] and E[3^{e3}] ,
where 2^{e2} ≈ 3^{e3}

Select “starting” curve: y^2 = x^3 + $ax + b$ over F_{p^2}

with characteristic p = $2^{e2}3^{e3}$ ± 1
such that #E = $(2^{e2}3^{e3})^2$ i.e. has E[2^{e2}] and E[3^{e3}]

Fix base points:
P_a and Q_a of E[2^{e2}] - basis of Alice
P_b and Q_b of E[3^{e3}] - basis of Bob

SIDH (Supersingular Isogeny Diffie-Hellman)
D. Jao and L. De Feo, 2011

Fixed public parameters:

$y^2 = x^3 + ax + b$ over F_{p^2}

$\{P_a , Q_a\}$ - basis of E$[2^{e2}]$
$\{P_b , Q_b\}$ - basis of E$[3^{e3}]$

SIDH (Supersingular Isogeny Diffie-Hellman)
D. Jao and L. De Feo, 2011

Alice generates key pair:

picks up random private key a : $0 < a < 2^{e2}$
kernel group generator $G_a = P_a + a * Q_a$
calculates isogeny φ_a with kernel group generated by G_a :
$E_a = E/< G_a >$
maps Bob's basis $\{P_b , Q_b\}$ to curve E_a : $\{\varphi_a(P_b) , \varphi_a(Q_b)\}$
sends to Bob her public key :

$$E_a,\ \varphi_a(P_b),\ \varphi_a(Q_b)$$

SIDH (Supersingular Isogeny Diffie-Hellman) D. Jao and L. De Feo, 2011

Upon receiving public key of Alice, Bob generates key pair:

picks up random private key b : $0 < b < 3^{e3}$
$G_b = P_b + b * Q_b$: point of order 3^{e3}

calculates isogeny φ_b with kernel group generated by G_b :
$E_b = E/< G_b >$
maps Alice's basis $\{P_a , Q_a\}$ to curve E_b : $\{\varphi_b(P_a) , \varphi_b(Q_a)\}$
sends to Alice his public key :

$$E_b,\ \varphi_b(P_a),\ \varphi_b(Q_a)$$

SIDH (Supersingular Isogeny Diffie-Hellman) D. Jao and L. De Feo, 2011

Bob:

$$G_{ba} = \varphi_a(P_b) + b*\varphi_a(Q_b)$$

$$E_{ba} = E_a/< G_{ba} >$$

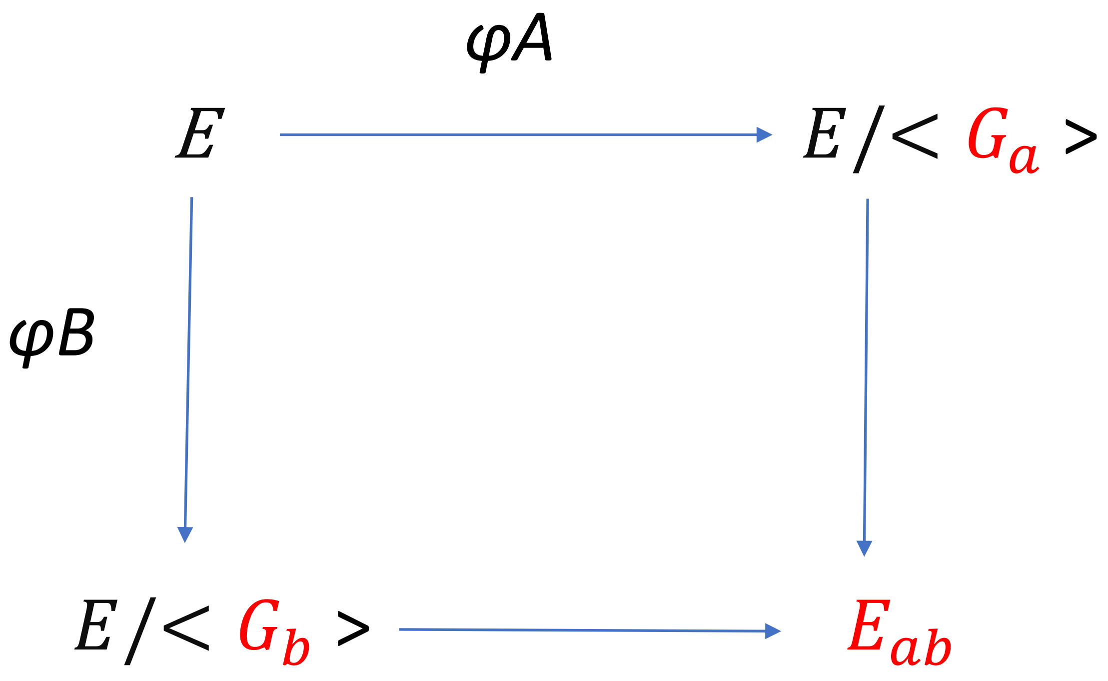
Alice:

$$G_{ab} = \varphi_b(P_a) + a*\varphi_b(Q_a)$$

$$E_{ab} = E_b/< G_{ab} >$$

Shared secret : $j\ (E_{ba}) = j\ (E_{ab})$

Commutative diagram



$$E_{ab} = E/< G_b >/< \varphi B(G_a) > \; = \; E/< G_a >/< \varphi A(G_b) >$$

Our solution to the problem of postquantum PAKE

Towards Isogeny-Based Password-Authenticated Key Establishment

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Abstract. Password authenticated key establishment (PAKE) is a cryptographic primitive that allows two parties who share a low-entropy secret (a password) to securely establish cryptographic keys in the absence of public key infrastructure. We propose the first quantum-resistant password-authenticated key exchange scheme based on supersingular elliptic curve isogenies. The scheme is built upon supersingular isogeny Diffie-Hellman [15], and uses the password to generate permutations which obscure the auxiliary points. We include elements of a security proof, and discuss roadblocks to obtaining a proof in the BPR model [1]. We also include some performance results.

From SIDH to SIDH PAKE

Ephemeral public key of Alice:

$$E_a,\ \varphi_a(P_b),\ \varphi_a(Q_b)$$

Alice calculates masked public key:

$$MaskedP = MaskP + \varphi_a(P_b),\ \ MaskedQ = MaskQ + \varphi_a(Q_b)$$

(where $MaskP$ = F (“1”|| password) , $MaskQ$ = F (“2”|| password))

Alice sends to Bob $E_a,\ MaskedP,\ MaskedQ$

From SIDH to SIDH PAKE

Bob :

receives E_a, $MaskedP$, $MaskedQ$

calculates $\varphi_a(P_b)$ = $MaskedP - MaskP$,

$\varphi_a(Q_b)$ = $MaskedQ$ - $MaskQ$

and get Alice's public key : E_a, $\varphi_a(P_b)$, $\varphi_a(Q_b)$

Offline dictionary attack

Tate pairing

$e(P_b, Q_b)^{\deg(\varphi_a)}$ = $e(\varphi_a(P_b), \varphi_a(Q_b))$

$(\deg(\varphi_a) = 2^{e2})$

Attacker has E_a, $MaskedP$, $MaskedQ$

Calculates $MaskP_i$ and $MaskQ_i$ for candidates on password

If $e(P_b, Q_b)^{\deg(\varphi_a)}$ = $e(MaskedP - MaskP_i, MaskedQ - MaskQ_i)$

Then password is found (with high probability)

Möbius Action

$SL_2(l,e)$ = { Ψ ∈ $(Z/l^e Z)^{2\times 2}$: $\det(\mathrm{A})$ = 1 mod l^e}
$\Upsilon_2(l,e)$ = {Ψ ∈ $SL_2(l,e)$: A is upper triangular mod l}

$\Upsilon_2(l,e)$ acts on E[l^e] × E[l^e] like matrix-vector multiplication:

i.e. if Ψ = $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ then Ψ $\binom{X}{Y}$ = $\begin{pmatrix} \alpha * X + \beta * Y \\ \gamma * X + \delta * Y \end{pmatrix}$

SIDH PAKE

Alice masks her ephemeral public key $E_a,\ \varphi_a(P_b),\ \varphi_a(Q_b)$:

$$\begin{pmatrix} X_a \\ Y_a \end{pmatrix} = \Psi_A \begin{pmatrix} \varphi_a(P_b) \\ \varphi_a(Q_b) \end{pmatrix}$$

Ψ_A is a function of password and j-invariant of E_a

Sends E_a, X_a, Y_a to Bob

Bob, upon receiving E_a, X_a, Y_a :

checks that $e(P_b, Q_b)^{\deg(\varphi_a)} == e(X_a\ ,\ Y_a)$ - if not, abort

SIDH PAKE

If pairing check is ok:

Bob unmasks masked ephemeral public key E_a, X_a, Y_a :

calculates inverse Ψ_A^{-1} from matrix $\Psi_A = H_A$ (password, $j(E_a)$)

restores ephemeral public key :

$$\begin{pmatrix} \varphi_a(P_b) \\ \varphi_a(Q_b) \end{pmatrix} = \Psi_A^{-1} \begin{pmatrix} X_a \\ Y_a \end{pmatrix}$$

And obtains “clear” SIDH ephemeral public key $E_a, \varphi_a(P_b), \varphi_a(Q_b)$

SIDH PAKE

Bob generates his key pair :

picks up random private key b : $0 < b < 3^{e3}$

calculates public key:

$E_b = E/< P_b + b * Q_b >, \ \varphi_b(P_a), \ \varphi_b(Q_a)$

masks his public key:

$\Psi_B = H_B(\text{password}, j(E_b))$

$$\begin{pmatrix} X_b \\ Y_b \end{pmatrix} = \Psi_B \begin{pmatrix} \varphi_b(P_a) \\ \varphi_b(Q_a) \end{pmatrix}$$

sends E_b, X_b, Y_b to Alice

SIDH PAKE

Calculates shared secret:

$$E_{ba} = E_a / < \varphi_a(P_b) + b * \varphi_a(Q_b) >$$

Shared secret :

KDF ((E_a, X_a, Y_a) || (E_b, X_b, Y_b) || $j(E_{ba})$ || Ψ_A || Ψ_B)

SIDH PAKE

Upon receiving E_b , X_b , Y_b from Bob, Alice:

checks that $e(P_a, Q_a)^{\deg(\varphi_b)} == e(X_b , Y_b)$ - if not, abort

demasks : $\begin{pmatrix} \varphi_b(P_a) \\ \varphi_b(Q_a) \end{pmatrix} = \Psi_B^{-1} \begin{pmatrix} X_b \\ Y_b \end{pmatrix}$

$E_{ab} = E_b / < \varphi_b(P_a) + a * \varphi_b(Q_a) >$

Shared secret:

KDF ((E_a , X_a , Y_a) || (E_b , X_b , Y_b) || $j\ (E_{ab})$ || Ψ_A || Ψ_B)

Practical aspects

Curves:

from SIKE algorithm (now in a second round of NIST Post-Quantum Cryptography Standardization Process)

Ephemeral key sizes:

just the same as in SIDH
(for SIKE's curves p434 and p503 : 330 and 378 bytes resp.)

Time:

from 1,7 to 2 of "pure" SIDH:
(for SIKE's curves p434 and p503 : 142 and 228 of 10^6 clock cycles resp.
Ubuntu 18.04, 1.6 GHz Intel Core i5-8250U)

Questions ?